Hilbert Space Methods for Partial Differential Equations

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Preface

This book is an outgrowth of a course which we have given almost periodically over the last eight years. It is addressed to beginning graduate students of mathematics, engineering, and the physical sciences. Thus, we have attempted to present it while presupposing a minimal background: the reader is assumed to have some prior acquaintance with the concepts of "linear" and "continuous" and also to believe L^2 is complete. An undergraduate mathematics training through Lebesgue integration is an ideal background but we dare not assume it without turning away many of our best students. The formal prerequisite consists of a good advanced calculus course and a motivation to study partial differential equations.

A problem is called *well-posed* if for each set of data there exists exactly one solution and this dependence of the solution on the data is continuous. To make this precise we must indicate the space from which the solution is obtained, the space from which the data may come, and the corresponding notion of continuity. Our goal in this book is to show that various types of problems are well-posed. These include boundary value problems for (stationary) elliptic partial differential equations and initial-boundary value problems for (time-dependent) equations of parabolic, hyperbolic, and pseudo-parabolic types. Also, we consider some nonlinear elliptic boundary value problems, variational or uni-lateral problems, and some methods of numerical approximation of solutions.

We briefly describe the contents of the various chapters. Chapter I presents all the elementary Hilbert space theory that is needed for the book. The first half of Chapter I is presented in a rather brief fashion and is intended both as a review for some readers and as a study guide for others. Non-standard items to note here are the spaces $C^m(\bar{G})$, V^*, and V'. The first consists of restrictions to the closure of G of functions on $\mathbb{R}^n$ and the last two consist of conjugate-linear functionals.

Chapter II is an introduction to distributions and Sobolev spaces. The latter are the Hilbert spaces in which we shall show various problems are well-posed. We use a primitive (and non-standard) notion of distribution which is adequate for our purposes. Our distributions are conjugate-linear and have the pedagogical advantage of being independent of any discussion of topological vector space theory.

Chapter III is an exposition of the theory of linear elliptic boundary value problems in variational form. (The meaning of "variational form" is explained in Chapter VII.) We present an abstract Green's theorem which permits the separation of the abstract problem into a partial differential equation on the region and a condition on the boundary. This approach has the pedagogical advantage of making optional the discussion of regularity theorems. (We construct an operator ∂ which is an extension of the normal derivative on the boundary, whereas the normal derivative makes sense only for appropriately regular functions.)

Chapter IV is an exposition of the generation theory of linear semigroups of contractions and its applications to solve initial-boundary value problems for partial differential equations. Chapters V and VI provide the immediate extensions to cover evolution equations of second order and of implicit type. In addition to the classical heat and wave equations with standard boundary conditions, the applications in these chapters include a multitude of non-standard problems such as equations of pseudo-parabolic, Sobolev, viscoelasticity, degenerate or mixed type; boundary conditions of periodic or non-local type or with time-derivatives; and certain interface or even global constraints on solutions. We hope this variety of applications may arouse the interests even of experts.

Chapter VII begins with some reflections on Chapter III and develops into an elementary alternative treatment of certain elliptic boundary value problems by the classical Dirichlet principle. Then we briefly discuss certain unilateral boundary value problems, optimal control problems, and numerical approximation methods. This chapter can be read immediately after Chapter III and it serves as a natural place to begin work on nonlinear problems.

There are a variety of ways this book can be used as a text. In a year course for a well-prepared class, one may complete the entire book and supplement it with some related topics from nonlinear functional analysis. In a semester course for a class with varied backgrounds, one may cover Chapters I, II, III, and VII. Similarly, with that same class one could cover in one semester the first four chapters. In any abbreviated treatment one could omit I.6, II.4, II.5, III.6, the last three sections of IV, V, and VI, and VII.4. We have included over 40 examples in the exposition and there are about 200 exercises. The exercises are placed at the ends of the chapters and each is numbered so as to indicate the section for which it is appropriate.

Some suggestions for further study are arranged by chapter and precede the Bibliography. If the reader develops the interest to pursue some topic in one of these references, then this book will have served its purpose.

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Chapter I

Elements of Hilbert Space

1 Linear Algebra

We begin with some notation. A function F with domain $\mathrm{dom}(F) = A$ and range $\mathrm{Rg}(F)$ a subset of B is denoted by $F : A \to B$. That a point $x \in A$ is mapped by F to a point $F(x) \in B$ is indicated by $x \mapsto F(x)$. If S is a subset of A then the *image* of S by F is $F(S) = \{F(x) : x \in S\}$. Thus $\mathrm{Rg}(F) = F(A)$. The pre-image or inverse image of a set $T \subset B$ is $F^{-1}(T) = \{x \in A : F(x) \in T\}$. A function is called *injective* if it is one-to-one, *surjective* if it is onto, and *bijective* if it is both injective and surjective. Then it is called, respectively, an *injection, surjection*, or *bijection*.

$\mathbb{K}$ will denote the field of scalars for our vector spaces and is always one of $\mathbb{R}$ (real number system) or $\mathbb{C}$ (complex numbers). The choice in most situations will be clear from the context or immaterial, so we usually avoid mention of it.

The "strong inclusion" $K \subset\subset G$ between subsets of Euclidean space $\mathbb{R}^n$ means K is compact, G is open, and $K \subset G$. If A and B are sets, their Cartesian product is given by $A \times B = \{[a, b] : a \in A,\ b \in B\}$. If A and B are subsets of $\mathbb{K}^n$ (or any other vector space) their set sum is $A + B = \{a + b : a \in A,\ b \in B\}$.

1.1

A *linear space* over the field $\mathbb{K}$ is a non-empty set V of vectors with a binary operation *addition* $+ : V \times V \to V$ and a *scalar multiplication* $\cdot : \mathbb{K} \times V \to V$

such that $(V,+)$ is an Abelian group, i.e.,

$$\begin{array}{ll}
(x+y)+z = x+(y+z)\ , & x,y,z \in V\ , \\
\text{there is a zero } \theta \in V : x+\theta = x\ , & x \in V\ , \\
\text{if } x \in V, \text{ there is } -x \in V : x+(-x) = \theta\ , & \text{and} \\
x+y = y+x\ , & x,y \in V\ ,
\end{array}$$

and we have

$$\begin{array}{lll}
(\alpha+\beta)\cdot x = \alpha\cdot x + \beta\cdot x\ , & \alpha\cdot(x+y) = \alpha\cdot x + \alpha\cdot y\ , & \\
\alpha\cdot(\beta\cdot x) = (\alpha\beta)\cdot x\ , & 1\cdot x = x\ , & x,y \in V\ ,\ \alpha,\beta \in \mathbb{K}\ .
\end{array}$$

We shall suppress the symbol for scalar multiplication since there is no need for it.

Examples. (a) The set $\mathbb{K}^n$ of n-tuples of scalars is a linear space over $\mathbb{K}$. Addition and scalar multiplication are defined coordinatewise:

$$\begin{array}{l}
(x_1,x_2,\ldots,x_n)+(y_1,y_2,\ldots,y_n) = (x_1+y_1,x_2+y_2,\ldots,x_n+y_n) \\
\alpha(x_1,x_2,\ldots,x_n) = (\alpha x_1,\alpha x_2,\ldots,\alpha x_n)\ .
\end{array}$$

(b) The set $\mathbb{K}^X$ of functions $f : X \to \mathbb{K}$ is a linear space, where X is a non-empty set, and we define $(f_1+f_2)(x) = f_1(x)+f_2(x)$, $(\alpha f)(x) = \alpha f(x)$, $x \in X$.

(c) Let $G \subset \mathbb{R}^n$ be open. The above pointwise definitions of linear operations give a linear space structure on the set $C(G,\mathbb{K})$ of continuous $f : G \to \mathbb{K}$. We normally shorten this to $C(G)$.

(d) For each n-tuple $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n)$ of non-negative integers, we denote by D^α the *partial derivative*

$$\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}$$

of order $|\alpha| = \alpha_1+\alpha_2+\cdots+\alpha_n$. The sets $C^m(G) = \{f \in C(G) : D^\alpha f \in C(G) \text{ for all } \alpha,\ |\alpha| \le m\}$, $m \ge 0$, and $C^\infty G = \bigcap_{m\ge 1} C^m(G)$ are linear spaces with the operations defined above. We let D^θ be the identity, where $\theta = (0,0,\ldots,0)$, so $C^0(G) = C(G)$.

(e) For $f \in C(G)$, the *support* of f is the closure in G of the set $\{x \in G : f(x) \ne 0\}$ and we denote it by $\text{supp}(f)$. $C_0(G)$ is the subset of those functions in $C(G)$ with compact support. Similarly, we define $C_0^m(G) = C^m(G)\cap C_0(G)$, $m \ge 1$ and $C_0^\infty(G) = C^\infty(G)\cap C_0(G)$.

(f) If $f : A \to B$ and $C \subset A$, we denote $f|_C$ the *restriction* of f to C. We obtain useful linear spaces of functions on the closure $\bar{G}$ as follows:

$$C^m(\bar{G}) = \{f|_{\bar{G}} : f \in C_0^m(\mathbb{R}^n)\} \quad , \quad C^\infty(\bar{G}) = \{f|_{\bar{G}} : f \in C_0^\infty(\mathbb{R}^n)\} \ .$$

These spaces play a central role in our work below.

1.2

A subset M of the linear space V is a *subspace* of V if it is closed under the linear operations. That is, $x + y \in M$ whenever $x, y \in M$ and $\alpha x \in M$ for each $\alpha \in \mathbb{K}$ and $x \in M$. We denote that M is a subspace of V by $M \leq V$. It follows that M is then (and only then) a linear space with addition and scalar multiplication inherited from V.

Examples. We have three chains of subspaces given by

$$\begin{aligned} C^j(G) \leq C^k(G) &\ \leq\ \mathbb{K}^G \ , \\ C^j(\bar{G}) \leq C^k(\bar{G}) &\ , \quad \text{and} \\ \{\theta\} \ \leq\ C_0^j(G) \leq C_0^k(G) &\ , \quad 0 \leq k \leq j \leq \infty \ . \end{aligned}$$

Moreover, for each k as above, we can identify $\varphi \in C_0^k(G)$ with that $\Phi \in C^k(\bar{G})$ obtained by defining Φ to be equal to φ on G and zero on ∂G, the boundary of G. Likewise we can identify each $\Phi \in C^k(\bar{G})$ with $\Phi|_G \in C^K(G)$. These identifications are "compatible" and we have $C_0^k(G) \leq C^k(\bar{G}) \leq C^k(G)$.

1.3

We let M be a subspace of V and construct a corresponding *quotient space*. For each $x \in V$, define a *coset* $\hat{x} = \{y \in V : y - x \in M\} = \{x + m : m \in M\}$. The set $V/M = \{\hat{x} : x \in V\}$ is the *quotient set*. Any $y \in \hat{x}$ is a *representative* of the coset $\hat{x}$ and we clearly have $y \in \hat{x}$ if and only if $x \in \hat{y}$ if and only if $\hat{x} = \hat{y}$. We shall define addition of cosets by adding a corresponding pair of representatives and similarly define scalar multiplication. It is necessary to first verify that this definition is unambiguous.

Lemma *If $x_1, x_2 \in \hat{x}$, $y_1, y_2 \in \hat{y}$, and $\alpha \in \mathbb{K}$, then $(\widehat{x_1 + y_1}) = (\widehat{x_2 + y_2})$ and $(\widehat{\alpha x_1}) = (\widehat{\alpha x_2})$.*

The proof follows easily, since M is closed under addition and scalar multiplication, and we can define $\hat{x}+\hat{y} = \widehat{(x+y)}$ and $\alpha\hat{x} = \widehat{(\alpha x)}$. These operations make V/M a linear space.

Examples. (a) Let $V = \mathbb{R}^2$ and $M = \{(0, x_2) : x_2 \in \mathbb{R}\}$. Then V/M is the set of parallel translates of the x_2-axis, M, and addition of two cosets is easily obtained by adding their (unique) representatives on the x_1-axis.
(b) Take $V = C(G)$. Let $x_0 \in G$ and $M = \{\varphi \in C(G) : \varphi(x_0) = 0\}$. Write each $\varphi \in V$ in the form $\varphi(x) = (\varphi(x) - \varphi(x_0)) + \varphi(x_0)$. This representation can be used to show that V/M is essentially equivalent (isomorphic) to $\mathbb{K}$.
(c) Let $V = C(\bar{G})$ and $M = C_0(G)$. We can describe V/M as a space of "boundary values." To do this, begin by noting that for each $K \subset\subset G$ there is a $\psi \in C_0(G)$ with $\psi = 1$ on K. (Cf. Section II.1.1.) Then write a given $\varphi \in C(\bar{G})$ in the form

$$\varphi = (\varphi\psi) + \varphi(1-\psi) \ ,$$

where the first term belongs to M and the second equals φ in a neighborhood of ∂G.

1.4

Let V and W be linear spaces over $\mathbb{K}$. A function $T : V \to W$ is *linear* if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \ , \qquad \alpha, \beta \in \mathbb{K} \ , \ x, y \in V \ .$$

That is, linear functions are those which preserve the linear operations. An *isomorphism* is a linear bijection. The set $\{x \in V : Tx = 0\}$ is called the *kernel* of the (not necessarily linear) function $T : V \to W$ and we denote it by $K(T)$.

Lemma *If $T : V \to W$ is linear, then $K(T)$ is a subspace of V, $\mathrm{Rg}(T)$ is a subspace of W, and $K(T) = \{\theta\}$ if and only if T is an injection.*

Examples. (a) Let M be a subspace of V. The identity $i_M : M \to V$ is a linear injection $x \mapsto x$ and its range is M.
(b) The *quotient* map $q_M : V \to V/M$, $x \mapsto \hat{x}$, is a linear surjection with kernel $K(q_M) = M$.
(c) Let G be the open interval (a, b) in $\mathbb{R}$ and consider $D \equiv d/dx$: $V \to C(\bar{G})$, where V is a subspace of $C^1(\bar{G})$. If $V = C^1(\bar{G})$, then D is a linear surjection with $K(D)$ consisting of constant functions on $\bar{G}$. If $V = \{\varphi \in C^1(\bar{G}) :$

$\varphi(a) = 0\}$, then D is an isomorphism. Finally, if $V = \{\varphi \in C^1(\bar{G}) : \varphi(a) = \varphi(b) = 0\}$, then $\mathrm{Rg}(D) = \{\varphi \in C(\bar{G}) : \int_a^b \varphi = 0\}$.

Our next result shows how each linear function can be factored into the product of a linear injection and an appropriate quotient map.

Theorem 1.1 *Let $T : V \to W$ be linear and M be a subspace of $K(T)$. Then there is exactly one function $\widehat{T} : V/M \to W$ for which $\widehat{T} \circ q_M = T$, and $\widehat{T}$ is linear with* $\mathrm{Rg}(\widehat{T}) = \mathrm{Rg}(T)$. *Finally, $\widehat{T}$ is injective if and only if $M = K(T)$.*

Proof: If $x_1, x_2 \in \hat{x}$, then $x_1 - x_2 \in M \subset K(T)$, so $T(x_1) = T(x_2)$. Thus we can define a function as desired by the formula $\widehat{T}(\hat{x}) = T(x)$. The uniqueness and linearity of $\widehat{T}$ follow since q_M is surjective and linear. The equality of the ranges follows, since q_M is surjective, and the last statement follows from the observation that $K(T) \subset M$ if and only if $v \in V$ and $\widehat{T}(\hat{x}) = 0$ imply $\hat{x} = \hat{0}$.

An immediate corollary is that each linear function $T : V \to W$ can be factored into a product of a surjection, an isomorphism, and an injection: $T = i_{\mathrm{Rg}(T)} \circ \widehat{T} \circ q_{K(T)}$.

A function $T : V \to W$ is called *conjugate linear* if

$$T(\alpha x + \beta y) = \bar{\alpha} T(x) + \bar{\beta} T(y) \ , \qquad \alpha, \beta \in \mathbb{K} \ , \ x, y \in V \ .$$

Results similar to those above hold for such functions.

1.5

Let V and W be linear spaces over $\mathbb{K}$ and consider the set $L(V, W)$ of linear functions from V to W. The set W^V of all functions from V to W is a linear space under the pointwise definitions of addition and scalar multiplication (cf. Example 1.1(b)), and $L(V, W)$ is a subspace.

We define V^* to be the linear space of all conjugate linear functionals from $V \to \mathbb{K}$. V^* is called the *algebraic dual* of V. Note that there is a bijection $f \mapsto \bar{f}$ of $\mathcal{L}(V, \mathbb{K})$ onto V^*, where $\bar{f}$ is the functional defined by $\bar{f}(x) = \overline{f(x)}$ for $x \in V$ and is called the *conjugate* of the functional $f : V \to \mathbb{K}$. Such spaces provide a useful means of constructing large linear spaces containing a given class of functions. We illustrate this technique in a simple situation.

Example. Let G be open in $\mathbb{R}^n$ and $x_0 \in G$. We shall imbed the space $C(G)$ in the algebraic dual of $C_0(G)$. For each $f \in C(G)$, define $T_f \in C_0(G)^*$ by

$$T_f(\varphi) = \int_G f\bar{\varphi} , \qquad \varphi \in C_0(G) .$$

Since $f\bar{\varphi} \in C_0(G)$, the Riemann integral is adequate here. An easy exercise shows that the function $f \mapsto T_f : C(G) \to C_0(G)^*$ is a linear injection, so we may thus identify $C(G)$ with a subspace of $C_0(G)^*$. This linear injection is not surjective; we can exhibit functionals on $C_0(G)$ which are not identified with functions in $C(G)$. In particular, the *Dirac functional* δ_{x_0} defined by

$$\delta_{x_0}(\varphi) = \overline{\varphi(x_0)} , \qquad \varphi \in C_0(G) ,$$

cannot be obtained as T_f for any $f \in C(G)$. That is, $T_f = \delta_{x_0}$ implies that $f(x) = 0$ for all $x \in G$, $x \neq x_0$, and thus $f = 0$, a contradiction.

2 Convergence and Continuity

The absolute value function on $\mathbb{R}$ and modulus function on $\mathbb{C}$ are denoted by $|\cdot|$, and each gives a notion of length or distance in the corresponding space and permits the discussion of convergence of sequences in that space or continuity of functions on that space. We shall extend these concepts to a general linear space.

2.1

A *seminorm* on the linear space V is a function $p : V \to \mathbb{R}$ for which $p(\alpha x) = |\alpha| p(x)$ and $p(x + y) \leq p(x) + p(y)$ for all $\alpha \in \mathbb{K}$ and $x, y \in V$. The pair V, p is called a *seminormed space*.

Lemma 2.1 *If V, p is a seminormed space, then*

(a) $|p(x) - p(y)| \leq p(x - y)$, $\qquad x, y \in V$,

(b) $p(x) \geq 0$, $\qquad x \in V$, *and*

(c) *the kernel $K(p)$ is a subspace of V.*

(d) *If $T \in L(W, V)$, then $p \circ T : W \to \mathbb{R}$ is a seminorm on W.*

(e) *If* p_j *is a seminorm on* V *and* $\alpha_j \geq 0$, $1 \leq j \leq n$, *then* $\sum_{j=1}^{n} \alpha_j p_j$ *is a seminorm on* V.

Proof: We have $p(x) = p(x-y+y) \leq p(x-y)+p(y)$ so $p(x)-p(y) \leq p(x-y)$. Similarly, $p(y) - p(x) \leq p(y-x) = p(x-y)$, so the result follows. Setting $y = 0$ in (a) and noting $p(0) = 0$, we obtain (b). The result (c) follows directly from the definitions, and (d) and (e) are straightforward exercises.

If p is a seminorm with the property that $p(x) > 0$ for each $x \neq \theta$, we call it a *norm*.

Examples. (a) For $1 \leq k \leq n$ we define seminorms on $\mathbb{K}^n$ by $p_k(x) = \sum_{j=1}^{k} |x_j|$, $q_k(x) = (\sum_{j=1}^{k} |x_j|^2)^{1/2}$, and $r_k(x) = \max\{|x_j| : 1 \leq j \leq k\}$. Each of p_n, q_n and r_n is a norm.

(b) If $J \subset X$ and $f \in \mathbb{K}^X$, we define $p_J(f) = \sup\{|f(x)| : x \in J\}$. Then for each finite $J \subset X$, p_J is a seminorm on $\mathbb{K}^X$.

(c) For each $K \subset\subset G$, p_K is a seminorm on $C(G)$. Also, $p_{\bar{G}} = p_G$ is a norm on $C(\bar{G})$.

(d) For each j, $0 \leq j \leq k$, and $K \subset\subset G$ we can define a seminorm on $C^k(G)$ by $p_{j,K}(f) = \sup\{|D^\alpha f(x)| : x \in K,\ |\alpha| \leq j\}$. Each such $p_{j,G}$ is a norm on $C^k(\bar{G})$.

2.2

Seminorms permit a discussion of convergence. We say the sequence $\{x_n\}$ in V *converges* to $x \in V$ if $\lim_{n\to\infty} p(x_n - x) = 0$; that is, if $\{p(x_n - x)\}$ is a sequence in $\mathbb{R}$ converging to 0. Formally, this means that for every $\varepsilon > 0$ there is an integer $N \geq 0$ such that $p(x_n - x) < \varepsilon$ for all $n \geq N$. We denote this by $x_n \to x$ in V, p and suppress the mention of p when it is clear what is meant.

Let $S \subset V$. The *closure* of S in V, p is the set $\bar{S} = \{x \in V : x_n \to x$ in V, p for some sequence $\{x_n\}$ in $S\}$, and S is called *closed* if $S = \bar{S}$. The closure $\bar{S}$ of S is the smallest closed set containing $S : S \subset \bar{S}$, $\bar{S} = \bar{\bar{S}}$, and if $S \subset K = \bar{K}$ then $\bar{S} \subset K$.

Lemma *Let* V, p *be a seminormed space and* M *be a subspace of* V. *Then* $\bar{M}$ *is a subspace of* V.

Proof: Let $x, y \in \bar{M}$. Then there are sequences $x_n, y_n \in M$ such that $x_n \to x$ and $y_n \to y$ in V, p. But $p((x+y)-(x_n+y_n)) \leq p(x-x_n)+p(y-y_n) \to$

0 which shows that $(x_n + y_n) \to x + y$. Since $x_n + y_n \in M$, all n, this implies that $x+y \in \bar{M}$. Similarly, for $\alpha \in \mathbb{K}$ we have $p(\alpha x - \alpha x_n) = |\alpha| p(x - x_n) \to 0$, so $\alpha x \in \bar{M}$.

2.3

Let V, p and W, q be seminormed spaces and $T : V \to W$ (not necessarily linear). Then T is called *continuous at* $x \in V$ if for every $\varepsilon > 0$ there is a $\delta > 0$ for which $y \in V$ and $p(x - y) < \delta$ implies $q(T(x) - T(y)) < \varepsilon$. T is *continuous* if it is continuous at every $x \in V$.

Theorem 2.2 *T is continuous at x if and only if $x_n \to x$ in V, p implies $Tx_n \to Tx$ in W, q.*

Proof: Let T be continuous at x and $\varepsilon > 0$. Choose $\delta > 0$ as in the definition above and then N such that $n \geq N$ implies $p(x_n - x) < \delta$, where $x_n \to x$ in V, p is given. Then $n \geq N$ implies $q(Tx_n - Tx) < \varepsilon$, so $Tx_n \to Tx$ in W, q.

Conversely, if T is not continuous at x, then there is an $\varepsilon > 0$ such that for every $n \geq 1$ there is an $x_n \in V$ with $p(x_n - x) < 1/n$ and $q(Tx_n - Tx) \geq \varepsilon$. That is, $x_n \to x$ in V, p but $\{Tx_n\}$ does not converge to Tx in W, q.

We record the facts that our algebraic operations and seminorm are always continuous.

Lemma *If V, p is a seminormed space, the functions $(\alpha, x) \mapsto \alpha x : \mathbb{K} \times V \to V$, $(x, y) \mapsto x + y : V \times V \to V$, and $p : V \to \mathbb{R}$ are all continuous.*

Proof: The estimate $p(\alpha x - \alpha_n x_n) \leq |\alpha - \alpha_n| p(x) + |\alpha_n| p(x - x_n)$ implies the continuity of scalar multiplication. Continuity of addition follows from an estimate in the preceding Lemma, and continuity of p follows from the Lemma of 2.1.

Suppose p and q are seminorms on the linear space V. We say p is *stronger* than q (or q is *weaker* than p) if for any sequence $\{x_n\}$ in V, $p(x_n) \to 0$ implies $q(x_n) \to 0$.

Theorem 2.3 *The following are equivalent:*

(a) *p is stronger than q,*

(b) *the identity* $I : V, p \to V, q$ *is continuous, and*

(c) *there is a constant* $K \geq 0$ *such that*

$$q(x) \leq Kp(x) \ , \qquad x \in V \ .$$

Proof: By Theorem 2.2, (a) is equivalent to having the identity $I : V, p \to V, q$ continuous at 0, so (b) implies (a). If (c) holds, then $q(x - y) \leq Kp(x - y)$, $x, y \in V$, so (b) is true.

We claim now that (a) implies (c). If (c) is false, then for every integer $n \geq 1$ there is an $x_n \in V$ for which $q(x_n) > np(x_n)$. Setting $y_n = (1/q(x_n))x_n$, $n \geq 1$, we have obtained a sequence for which $q(y_n) = 1$ and $p(y_n) \to 0$, thereby contradicting (a).

Theorem 2.4 *Let* V, p *and* W, q *be seminormed spaces and* $T \in L(V, W)$. *The following are equivalent:*

(a) T *is continuous at* $\theta \in V$,

(b) T *is continuous, and*

(c) *there is a constant* $K \geq 0$ *such that*

$$q(T(x)) \leq Kp(x) \ , \qquad x \in V \ .$$

Proof: By Theorem 2.3, each of these is equivalent to requiring that the seminorm p be stronger than the seminorm $q \circ T$ on V.

2.4

If V, p and W, q are seminormed spaces, we denote by $\mathcal{L}(V, W)$ the set of continuous linear functions from V to W. This is a subspace of $L(V, W)$ whose elements are frequently called the *bounded* operators from V to W (because of Theorem 2.4).

Let $T \in \mathcal{L}(V, W)$ and consider

$$\begin{aligned} \lambda &\equiv \sup\{q(T(x)) : x \in V \ , \quad p(x) \leq 1\} \ , \\ \mu &\equiv \inf\{K > 0 : q(T(x)) \leq Kp(x) \ \text{ for all } \ x \in V\} \ . \end{aligned}$$

If K belongs to the set defining μ, then for every $x \in V : p(x) \leq 1$ we have $q(T(x)) \leq K$, hence $\lambda \leq K$. This holds for all such K, so $\lambda \leq \mu$. If $x \in V$ with $p(x) > 0$, then $y \equiv (1/p(x))x$ satisfies $p(y) = 1$, so $q(T(y)) \leq \lambda$. That is $q(T(x)) \leq \lambda p(x)$ whenever $p(x) > 0$. But by Theorem 2.4(c) this last inequality is trivially satisfied when $p(x) = 0$, so we have $\mu \leq \lambda$. These remarks prove the first part of the following result; the remaining parts are straightforward.

Theorem 2.5 *Let V, p and W, q be seminormed spaces. For each $T \in \mathcal{L}(V, W)$ we define a real number by $|T|_{p,q} \equiv \sup\{q(T(x)) : x \in V,\ p(x) \leq 1\}$. Then we have $|T|_{p,q} = \sup\{q(T(x)) : x \in V,\ p(x) = 1\} = \inf\{K > 0 : q(T(x)) \leq Kp(x)$ for all $x \in V\}$ and $|\cdot|_{p,q}$ is a seminorm on $\mathcal{L}(V, W)$. Furthermore, $q(T(x)) \leq |T|_{p,q} \cdot p(x)$, $x \in V$, and $|\cdot|_{p,q}$ is a norm whenever q is a norm.*

Definitions. The *dual* of the seminormed space V, p is the linear space $V' = \{f \in V^* : f$ is continuous$\}$ with the norm

$$\|f\|_{V'} = \sup\{|f(x)| : x \in V\ ,\quad p(x) \leq 1\}\ .$$

If V, p and W, q are seminormed spaces, then $T \in \mathcal{L}(V, W)$ is called a *contraction* if $|T|_{p,q} \leq 1$, and T is called an *isometry* if $|T|_{p,q} = 1$.

3 Completeness

3.1

A sequence $\{x_n\}$ in a seminormed space V, p is called *Cauchy* if $\lim_{m,n\to\infty} p(x_m - x_n) = 0$, that is, if for every $\varepsilon > 0$ there is an integer N such that $p(x_m - x_n) < \varepsilon$ for all $m, n \geq N$. Every convergent sequence is Cauchy. We call V, p *complete* if every Cauchy sequence is convergent. A complete normed linear space is a *Banach space*.

Examples. Each of the seminormed spaces of Examples 2.1(a-d) is complete.

(e) Let $G = (0, 1) \subset \mathbb{R}^1$ and consider $C(\bar{G})$ with the norm $p(x) = \int_0^1 |x(t)|\,dt$. Let $0 < c < 1$ and for each n with $0 < c - 1/n$ define $x_n \in C(\bar{G})$ by

$$x_n(t) = \begin{cases} 1\ , & c \leq t \leq 1 \\ n(t - c) + 1\ , & c - 1/n < t < c \\ 0\ , & 0 \leq t \leq c - 1/n \end{cases}$$

For $m \geq n$ we have $p(x_m - x_n) \leq 1/n$, so $\{x_m\}$ is Cauchy. If $x \in C(\bar{G})$, then

$$p(x_n - x) \geq \int_0^{c-1/n} |x(t)|\, dt + \int_c^1 |1 - x(t)|\, d(t)\ .$$

This shows that if $\{x_n\}$ converges to x then $x(t) = 0$ for $0 \leq t < c$ and $x(t) = 1$ for $c \leq t \leq 1$, a contradiction. Hence $C(\bar{G})$, p is not complete.

3.2

We consider the problem of extending a given function to a larger domain.

Lemma *Let $T : D \to W$ be given, where D is a subset of the seminormed space V, p and W, q is a normed linear space. There is at most one continuous $\bar{T} : \bar{D} \to W$ for which $\bar{T}|_D = T$.*

Proof: Suppose T_1 and T_2 are continuous functions from $\bar{D}$ to W which agree with T on D. Let $x \in \bar{D}$. Then there are $x_n \in D$ with $x_n \to x$ in V, p. Continuity of T_1 and T_2 shows $T_1 x_n \to T_1 x$ and $T_2 x_n \to T_2 x$. But $T_1 x_n = T_2 x_n$ for all n, so $T_1 x = T_2 x$ by the uniqueness of limits in the normed space W, q.

Theorem 3.1 *Let $T \in \mathcal{L}(D, W)$, where D is a subspace of the seminormed space V, p and W, q is a Banach space. Then there exists a unique $\bar{T} \in \mathcal{L}(\bar{D}, W)$ such that $\bar{T}|_D = T$, and $|\bar{T}|_{p,q} = |T|_{p,q}$.*

Proof: Uniqueness follows from the preceding lemma. Let $x \in \bar{D}$. If $x_n \in D$ and $x_n \to x$ in V, p, then $\{x_n\}$ is Cauchy and the estimate

$$q(T(x_m) - T(x_n)) \leq K p(x_m - x_n)$$

shows $\{T(x_n)\}$ is Cauchy in W, q, hence, convergent to some $y \in W$. If $x_n' \in D$ and $x_n' \to x$ in V, p, then $T x_n' \to y$, so we can define $\bar{T} : \bar{D} \to W$ by $T(x) = y$. The linearity of T on D and the continuity of addition and scalar multiplication imply that $\bar{T}$ is linear. Finally, the continuity of seminorms and the estimates

$$q(T(x_n)) \leq |T|_{p,q}\, p(x_n)$$

show $\bar{T}$ is continuous on $|\bar{T}|_{p,q} = |T|_{p,q}$.

3.3

A *completion* of the seminormed space V, p is a complete seminormed space W, q and a linear injection $T : V \to W$ for which $\mathrm{Rg}(T)$ is dense in W and T preserves seminorms: $q(T(x)) = p(x)$ for all $x \in V$. By identifying V, p with $\mathrm{Rg}(T), q$, we may visualize V as being dense and contained in a corresponding space that is complete. The completion of a normed space is a Banach space and linear injection as above. If two Banach spaces are completions of a given normed space, then we can use Theorem 3.1 to construct a linear norm-preserving bijection between them, so the completion of a normed space is essentially unique.

We first construct a completion of a given seminormed space V, p. Let W be the set of all Cauchy sequences in V, p. From the estimate $|p(x_n) - p(x_m)| \leq p(x_n - x_m)$ it follows that $\bar{p}(\{x_n\}) = \lim_{n\to\infty} p(x_n)$ defines a function $\bar{p} : W \to \mathbb{R}$ and it easily follows that $\bar{p}$ is a seminorm on W. For each $x \in V$, let $Tx = \{x, x, x, \ldots\}$, the indicated constant sequence. Then $T : V, p \to W, \bar{p}$ is a linear seminorm-preserving injection. If $\{x_n\} \in W$, then for any $\varepsilon > 0$ there is an integer N such that $p(x_n - x_N) < \varepsilon/2$ for $n \geq N$, and we have $\bar{p}(\{x_n\} - T(x_N)) \leq \varepsilon/2 < \varepsilon$. Thus, $\mathrm{Rg}(T)$ is dense in W. Finally, we verify that $W, \bar{p}$ is complete. Let $\{\bar{x}_n\}$ be a Cauchy sequence in $W, \bar{p}$ and for each $n \geq 1$ pick $x_n \in V$ with $\bar{p}(\bar{x}_n - T(x_n)) < 1/n$. Define $\bar{x}_0 = \{x_1, x_2, x_2, \ldots\}$. From the estimate

$$p(x_m - x_n) = \bar{p}(Tx_m - Tx_n) \leq 1/m + \bar{p}(\bar{x}_m - \bar{x}_n) + 1/n$$

it follows that $\bar{x}_0 \in W$, and from

$$\bar{p}(\bar{x}_n - \bar{x}_0) \leq \bar{p}(\bar{x}_n - Tx_n) + \bar{p}(Tx_n - \bar{x}_0) < 1/n + \lim_{m\to\infty} p(x_n - x_m)$$

we deduce that $\bar{x}_n \to \bar{x}_0$ in $W, \bar{p}$. Thus, we have proved the following.

Theorem 3.2 *Every seminormed space has a completion.*

3.4

In order to obtain from a normed space a corresponding normed completion, we shall identify those elements of W which have the same limit by factoring W by the kernel of $\bar{p}$. Before describing this quotient space, we consider quotients in a seminormed space.

Theorem 3.3 *Let V, p be a seminormed space, M a subspace of V and define*

$$\hat{p}(\hat{x}) = \inf\{p(y) : y \in \hat{x}\}\ , \qquad \hat{x} \in V/M\ .$$

(a) *$V/M, \hat{p}$ is a seminormed space and the quotient map $q : V \to V/M$ has $(p, \hat{p})$-seminorm $= 1$.*

(b) *If D is dense in V, then $\hat{D} = \{\hat{x} : x \in D\}$ is dense in V/M.*

(c) *$\hat{p}$ is a norm if and only if M is closed.*

(d) *If V, p is complete, then $V/M, \hat{p}$ is complete.*

Proof: We leave (a) and (b) as exercises. Part (c) follows from the observation that $\hat{p}(\hat{x}) = 0$ if and only if $x \in \bar{M}$.

To prove (d), we recall that a Cauchy sequence converges if it has a convergent subsequence so we need only consider a sequence $\{\hat{x}_n\}$ in V/M for which $\hat{p}(\hat{x}_{n+1} - \hat{x}_n) < 1/2^n$, $n \geq 1$. For each $n \geq 1$ we pick $y_n \in \hat{x}_n$ with $p(y_{n+1} - y_n) < 1/2^n$. For $m \geq n$ we obtain

$$p(y_m - y_n) \leq \sum_{k=0}^{m-1-n} p(y_{n+1+k} - y_{n+k}) < \sum_{k=0}^{\infty} 2^{-(n+k)} = 2^{1-n}\ .$$

Thus $\{y_n\}$ is Cauchy in V, p and part (a) shows $\hat{x}_n \to \hat{x}$ in V/M, where x is the limit of $\{y_n\}$ in V, p.

Given V, p and the completion $W, \bar{p}$ constructed for Theorem 3.2, we consider the quotient space W/K and its corresponding seminorm $\hat{p}$, where K is the kernel of $\bar{p}$. The continuity of $\bar{p} : W \to \mathbb{R}$ implies that K is closed, so $\hat{p}$ is a norm on W/K. $W, \bar{p}$ is complete, so W/K, $\hat{p}$ is a Banach space. The quotient map $q : W \to W/K$ satisfies $\hat{p}(q(x)) = \hat{p}(\hat{x}) = \bar{p}(y)$ for all $y \in q(x)$, so q preserves the seminorms. Since $\mathrm{Rg}(T)$ is dense in W it follows that the linear map $q \circ T : V \to W/K$ has a dense range in W/K. We have $\hat{p}((q \circ T)x) = \hat{p}(\widehat{Tx}) = p(x)$ for $x \in V$, hence $K(q \circ T) \leq K(p)$. If p is a norm this shows that $q \circ T$ is injective and proves the following.

Theorem 3.4 *Every normed space has a completion.*

3.5

We briefly consider the vector space $\mathcal{L}(V,W)$.

Theorem 3.5 *If V,p is a seminormed space and W,q is a Banach space, then $\mathcal{L}(V,W)$ is a Banach space. In particular, the dual V' of a seminormed space is complete.*

Proof: Let $\{T_n\}$ be a Cauchy sequence in $\mathcal{L}(V,W)$. For each $x \in V$, the estimate

$$q(T_m x - T_n x) \leq |T_m - T_n| p(x)$$

shows that $\{T_n x\}$ is Cauchy, hence convergent to a unique $T(x) \in W$. This defines $T : V \to W$ and the continuity of addition and scalar multiplication in W will imply that $T \in L(V,W)$. We have

$$q(T_n(x)) \leq |T_n| p(x) \ , \qquad x \in V \ ,$$

and $\{|T_n|\}$ is Cauchy, hence, bounded in $\mathbb{R}$, so the continuity of q shows that $T \in \mathcal{L}(V,W)$ with $|T| \leq K \equiv \sup\{|T_n| : n \geq 1\}$.

To show $T_n \to T$ in $\mathcal{L}(V,W)$, let $\varepsilon > 0$ and choose N so large that $m,n \geq N$ implies $|T_m - T_n| < \varepsilon$. Hence, for $m,n \geq N$, we have

$$q(T_m(x) - T_n(x)) < \varepsilon p(x) \ , \qquad x \in V \ .$$

Letting $m \to \infty$ shows that for $n \geq N$ we have

$$q(T(x) - T_n(x)) \leq \varepsilon p(x) \ , \qquad x \in V \ ,$$

so $|T - T_n| \leq \varepsilon$.

4 Hilbert Space

4.1

A *scalar product* on the vector space V is a function $V \times V \to \mathbb{K}$ whose value at x,y is denoted by (x,y) and which satisfies (a) $x \mapsto (x,y) : V \to \mathbb{K}$ is linear for every $y \in V$, (b) $(x,y) = \overline{(y,x)}$, $x,v \in V$, and (c) $(x,x) > 0$ for each $x \neq 0$. From (a) and (b) it follows that for each $x \in V$, the function $y \mapsto (x,y)$ is conjugate-linear, i.e., $(x,\alpha y) = \bar{\alpha}(x,y)$. The pair $V,(\cdot,\cdot)$ is called a *scalar product space*.

Theorem 4.1 *If $V, (\cdot,\cdot)$ is a scalar product space, then*

(a) $|(x,y)|^2 \leq (x,x)\cdot(y,y)$, $\qquad x,y \in V$,

(b) $\|x\| \equiv (x,x)^{1/2}$ *defines a norm* $\|\cdot\|$ *on* V *for which*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) , \qquad x,y \in V , \quad \textit{and}$$

(c) *the scalar product is continuous from* $V \times V$ *to* K.

Proof: Part (a) follows from the computation

$$0 \leq (\alpha x + \beta y, \alpha x + \beta y) = \beta(\beta(y,y) - |\alpha|^2)$$

for the scalars $\alpha = -\overline{(x,y)}$ and $\beta = (x,x)$. To prove (b), we use (a) to verify

$$\|x+y\|^2 \leq \|x\|^2 + 2|(x,y)| + \|y\|^2 \leq (\|x\| + \|y\|)^2 .$$

The remaining norm axioms are easy and the indicated identity is easily verified. Part (c) follows from the estimate

$$|(x,y) - (x_n,y_n)| \leq \|x\|\,\|y - y_n\| + \|y_n\|\,\|x - x_n\|$$

applied to a pair of sequences, $x_n \to x$ and $y_n \to y$ in $V, \|\cdot\|$.

A *Hilbert space* is a scalar product space for which the corresponding normed space is complete.

Examples. (a) Let $V = \mathbb{K}^N$ with vectors $x = (x_1, x_2, \ldots, x_N)$ and define $(x,y) = \sum_{j=1}^N x_j \bar{y}_j$. Then $V, (\cdot,\cdot)$ is a Hilbert space (with the norm $\|x\| = (\sum_{j=1}^N |x_j|^2)^{1/2}$) which we refer to as Euclidean space.

(b) We define $C_0(G)$ a scalar product by

$$(\varphi, \psi) = \int_G \varphi \bar{\psi}$$

where G is open in $\mathbb{R}^n$ and the Riemann integral is used. This scalar product space is not complete.

(c) On the space $L^2(G)$ of (equivalence classes of) Lebesgue square-summable $\mathbb{K}$-valued functions we define the scalar product as in (b) but with the Lebesgue integral. This gives a Hilbert space in which $C_0(G)$ is a dense subspace.

Suppose $V, (\cdot,\cdot)$ is a scalar product space and let $B, \|\cdot\|$ denote the completion of $V, \|\cdot\|$. For each $y \in V$, the function $x \mapsto (x,y)$ is linear, hence has a unique extension to B, thereby extending the definition of (x,y) to $B \times V$. It is easy to verify that for each $x \in B$, the function $y \mapsto (x,y)$ is in V' and we can similarly extend it to define (x,y) on $B \times B$. By checking that (the extended) function $(\cdot,\cdot)$ is a scalar product on B, we have proved the following result.

Theorem 4.2 *Every scalar product space has a (unique) completion which is a Hilbert space and whose scalar product is the extension by continuity of the given scalar product.*

Example. $L^2(G)$ is the completion of $C_0(G)$ with the scalar product given above.

4.2

The scalar product gives us a notion of angles between vectors. (In particular, recall the formula $(x,y) = \|x\|\,\|y\|\cos(\theta)$ in Example (a) above.) We call the vectors x, y *orthogonal* if $(x,y) = 0$. For a given subset M of the scalar product space V, we define the *orthogonal complement* of M to be the set

$$M^\perp = \{x \in V : (x,y) = 0 \text{ for all } y \in M\} \, .$$

Lemma *$M^\perp$ is a closed subspace of V and $M \cap M^\perp = \{0\}$.*

Proof: For each $y \in M$, the set $\{x \in V : (x,y) = 0\}$ is a closed subspace and so then is the intersection of all these for $y \in M$. The only vector orthogonal to itself is the zero vector, so the second statement follows.

A set K in the vector space V is *convex* if for $x, y \in K$ and $0 \le \alpha \le 1$, we have $\alpha x + (1-\alpha)y \in K$. That is, if a pair of vectors is in K, then so also is the line segment joining them.

Theorem 4.3 *A non-empty closed convex subset K of the Hilbert space H has an element of minimal norm.*

Proof: Setting $d \equiv \inf\{\|x\| : x \in K\}$, we can find a sequence $x_n \in K$ for which $\|x_n\| \to d$. Since K is convex we have $(1/2)(x_n + x_m) \in K$ for

$m, n \geq 1$, hence $\|x_n + x_m\|^2 \geq 4d^2$. From Theorem 4.1(b) we obtain the estimate $\|x_n - x_m\|^2 \leq 2(\|x_n\|^2 + \|x_m\|^2) - 4d^2$. The right side of this inequality converges to 0, so $\{x_n\}$ is Cauchy, hence, convergent to some $x \in H$. K is closed, so $x \in K$, and the continuity of the norm shows that $\|x\| = \lim_n \|x_n\| = d$.

We note that the element with minimal norm is unique, for if $y \in K$ with $\|y\| = d$, then $(1/2)(x + y) \in K$ and Theorem 4.1(b) give us, respectively, $4d^2 \leq \|x + y\|^2 = 4d^2 - \|x - y\|^2$. That is, $\|x - y\| = 0$.

Theorem 4.4 *Let M be a closed subspace of the Hilbert space H. Then for every $x \in H$ we have $x = m + n$, where $m \in M$ and $n \in M^\perp$ are uniquely determined by x.*

Proof: The uniqueness follows easily, since if $x = m_1 + n_1$ with $m_1 \in M$, $n_1 \in M^\perp$, then $m_1 - m = n - n_1 \in M \cap M^\perp = \{\theta\}$. To establish the existence of such a pair, define $K = \{x + y : y \in M\}$ and use Theorem 4.3 to find $n \in K$ with $\|n\| = \inf\{\|x + y\| : y \in M\}$. Then set $m = x - n$. It is clear that $m \in M$ and we need only to verify that $n \in M^\perp$. Let $y \in M$. For each $\alpha \in \mathbb{K}$, we have $n - \alpha y \in K$, hence $\|n - \alpha y\|^2 \geq \|n\|^2$. Setting $\alpha = \beta(n, y)$, $\beta > 0$, gives us $|(n, y)|^2(\beta\|y\|^2 - 2) \geq 0$, and this can hold for all $\beta > 0$ only if $(n, y) = 0$.

4.3

From Theorem 4.4 it follows that for each closed subspace M of a Hilbert space H we can define a function $P_M : H \to M$ by $P_M : x = m + n \mapsto m$, where $m \in M$ and $n \in M^\perp$ as above. The linearity of P_M is immediate and the computation

$$\|P_M x\|^2 \leq \|P_M x\|^2 + \|n\|^2 = \|P_M x + n\|^2 = \|x\|^2$$

shows $P_M \in \mathcal{L}(H, H)$ with $\|P_M\| \leq 1$. Also, $P_M x = x$ exactly when $x \in M$, so $P_M \circ P_M = P_M$. The operator P_M is called the *projection on M*.

If $P \in \mathcal{L}(B, B)$ satisfies $P \circ P = P$, then P is called a *projection* on the Banach space B. The result of Theorem 4.4 is a guarantee of a rich supply of projections in a Hilbert space.

4.4

We recall that the (continuous) dual of a seminormed space is a Banach space. We shall show there is a natural correspondence between a Hilbert space H and its dual H'. Consider for each fixed $x \in H$ the function f_x defined by the scalar product: $f_x(y) = (x, y)$, $y \in H$. It is easy to check that $f_x \in H'$ and $\|f_x\|_{H'} = \|x\|$. Furthermore, the map $x \mapsto f_x : H \to H'$ is linear:

$$\begin{aligned} f_{x+z} &= f_x + f_z \ , \qquad x, z \in H \ , \\ f_{\alpha x} &= \alpha f_x \ , \qquad \alpha \in \mathbb{K} \ , \ x \in H \ . \end{aligned}$$

Finally, the function $x \mapsto f_x : H \to H'$ is a norm preserving and linear injection. The above also holds in any scalar product space, but for Hilbert spaces this function is also surjective. This follows from the next result.

Theorem 4.5 *Let H be a Hilbert space and $f \in H'$. Then there is an element $x \in H$ (and only one) for which*

$$f(y) = (x, y) \ , \qquad y \in H \ .$$

Proof: We need only verify the existence of $x \in H$. If $f = \theta$ we take $x = \theta$, so assume $f \neq \theta$ in H'. Then the kernel of f, $K = \{x \in H : f(x) = 0\}$ is a closed subspace of H with $K^\perp \neq \{\theta\}$. Let $n \in K^\perp$ be chosen with $\|n\| = 1$. For each $z \in K^\perp$ it follows that $\overline{f(n)}z - \overline{f(z)}n \in K \cap K^\perp = \{\theta\}$, so z is a scalar multiple of n. (That is, $K^\perp$ is one-dimensional.) Thus, each $y \in H$ is of the form $y = P_K(y) + \lambda n$ where $(y, n) = \lambda(n, n) = \lambda$. But we also have $f(y) = \bar{\lambda} f(n)$, since $P_K(y) \in K$, and thus $f(y) = (f(n)n, y)$ for all $y \in H$.

The function $x \mapsto f_x$ from H to H' will occur frequently in our later discussions and it is called the *Riesz map* and is denoted by R_H. Note that it depends on the scalar product as well as the space. In particular, R_H is an isometry of H onto H' defined by

$$R_H(x)(y) = (x, y)_H \ , \qquad x, y \in H \ .$$

5 Dual Operators; Identifications

5.1

Suppose V and W are linear spaces and $T \in L(V,W)$. Then we define the *dual operator* $T' \in L(W^*,V^*)$ by

$$T'(f) = f \circ T \ , \qquad f \in W^* \ .$$

Theorem 5.1 *If V is a linear space, W,q is a seminorm space, and $T \in L(V,W)$ has dense range, then T' is injective on W'. If V,p and W,q are seminorm spaces and $T \in \mathcal{L}(V,W)$, then the restriction of the dual T' to W' belongs to $\mathcal{L}(W',V')$ and it satisfies*

$$\|T'\|_{\mathcal{L}(W',V')} \le |T|_{p,q} \ .$$

Proof: The first part follows from Section 3.2. The second is obtained from the estimate

$$|T'f(x)| \le \|f\|_{W'}|T|_{p,q}\, p(x) \ , \qquad f \in W' \ , \ x \in V \ .$$

We give two basic examples. Let V be a subspace of the seminorm space W,q and let $i : V \to W$ be the identity. Then $i'(f) = f \circ i$ is the restriction of f to the subspace V; i' is injective on W' if (and only if) V is dense in W. In such cases we may actually identify $i'(W')$ with W', and we denote this identification by $W' \le V^*$.

Consider the quotient map $q : W \to W/V$ where V and W,q are given as above. It is clear that if $g \in (W/V)^*$ and $f = q'(g)$, i.e., $f = g \circ q$, then $f \in W^*$ and $V \le K(f)$. Conversely, if $f \in W^*$ and $V \le K(f)$, then Theorem 1.1 shows there is a $g \in (W/V)^*$ for which $q'(g) = f$. These remarks show that $\mathrm{Rg}(q') = \{f \in W^* : V \le K(f)\}$. Finally, we note by Theorem 3.3 that $|q|_{q,\hat{q}} = 1$, so it follows that $g \in (W,V)'$ if and only if $q'(g) \in W'$.

5.2

Let V and W be Hilbert spaces and $T \in \mathcal{L}(V,W)$. We define the *adjoint* of T as follows: if $u \in W$, then the functional $v \mapsto (u,Tv)_W$ belongs to V', so Theorem 4.5 shows that there is a unique $T^*u \in V$ such that

$$(T^*u,v)_V = (u,Tv)_W \ , \qquad u \in W \ , \ v \in V \ .$$

Theorem 5.2 *If V and W are Hilbert spaces and $T \in \mathcal{L}(V,W)$, then $T^* \in \mathcal{L}(W,V)$, $\mathrm{Rg}(T)^\perp = K(T^*)$ and $\mathrm{Rg}(T^*)^\perp = K(T)$. If T is an isomorphism with $T^{-1} \in \mathcal{L}(W,V)$, then T^* is an isomorphism and $(T^*)^{-1} = (T^{-1})^*$.*

We leave the proof as an exercise and proceed to show that dual operators are essentially equivalent to the corresponding adjoint. Let V and W be Hilbert spaces and denote by R_V and R_W the corresponding Riesz maps (Section 4.4) onto their respective dual spaces. Let $T \in \mathcal{L}(V,W)$ and consider its dual $T' \in \mathcal{L}(W',V')$ and its adjoint $T^* \in \mathcal{L}(W,V)$. For $u \in W$ and $v \in V$ we have $R_V \circ T^*(u)(v) = (T^*u,v)_V = (u,Tv)_W = R_W(u)(Tv) = (T' \circ R_W u)(v)$. This shows that $R_V \circ T^* = T' \circ R_W$, so the Riesz maps permit us to study either the dual or the adjoint and deduce information on both. As an example of this we have the following.

Corollary 5.3 *If V and W are Hilbert spaces, and $T \in \mathcal{L}(V,W)$, then $\mathrm{Rg}(T)$ is dense in W if and only if T' is injective, and T is injective if and only if $\mathrm{Rg}(T')$ is dense in V'. If T is an isomorphism with $T^{-1} \in \mathcal{L}(W,V)$, then $T' \in \mathcal{L}(W',V')$ is an isomorphism with continuous inverse.*

5.3

It is extremely useful to make certain identifications between various linear spaces and we shall discuss a number of examples which will appear frequently in the following.

First, consider the linear space $C_0(G)$ and the Hilbert space $L^2(G)$. Elements of $C_0(G)$ are functions while elements of $L^2(G)$ are *equivalence classes* of functions. Since each $f \in C_0(G)$ is square-summable on G, it belongs to exactly one such equivalence class, say $i(f) \in L^2(G)$. This defines a linear injection $i : C_0(G) \to L^2(G)$ whose range is dense in $L^2(G)$. The dual $i' : L^2(G)' \to C_0(G)^*$ is then a linear injection which is just restriction to $C_0(G)$.

The Riesz map R of $L^2(G)$ (with the usual scalar product) onto $L^2(G)'$ is defined as in Section 4.4. Finally, we have a linear injection $T : C_0(G) \to C_0(G)^*$ given in Section 1.5 by

$$(Tf)(\varphi) = \int_G f(x)\bar{\varphi}(x)\,dx \ , \qquad f,\varphi \in C_0(G) \ .$$

Both R and T are possible identifications of (equivalence classes of) functions with conjugate-linear functionals. Moreover we have the important identity

$$T = i' \circ R \circ i \ .$$

This shows that all four injections may be used simultaneously to identify the various pairs as subspaces. That is, we identify

$$C_0(G) \leq L^2(G) = L^2(G)' \leq C_0(G)^* \ ,$$

and thereby reduce each of i, R, i' and T to the identity function from a subspace to the whole space. Moreover, once we identify $C_0(G) \leq L^2(G)$, $L^2(G)' \leq C_0(G)^*$, and $C_0(G) \leq C_0(G)^*$, by means of i, i', and T, respectively, then it follows that the identification of $L^2(G)$ with $L^2(G)'$ through the Riesz map R is possible (i.e., compatible with the three preceding) *only if* the R corresponds to the standard scalar product on $L^2(G)$. For example, suppose R is defined through the (equivalent) scalar-product

$$(Rf)(g) = \int_G a(x) f(x) \overline{g(x)}\, dx \ , \qquad f, g \in L^2(G) \ ,$$

where $a(\cdot) \in L^\infty(G)$ and $a(x) \geq c > 0$, $x \in G$. Then, with the three identifications above, R corresponds to multiplication by the function $a(\cdot)$. Other examples will be given later.

5.4

We shall find the concept of a sesquilinear form is as important to us as that of a linear operator. The theory of sesquilinear forms is analogous to that of linear operators and we discuss it briefly.

Let V be a linear space over the field $\mathbb{K}$. A *sesquilinear form* on V is a $\mathbb{K}$-valued function $a(\cdot,\cdot)$ on the product $V \times V$ such that $x \mapsto a(x,y)$ is linear for every $y \in V$ and $y \mapsto a(x,y)$ is conjugate linear for every $x \in V$. Thus, each sesquilinear form $a(\cdot,\cdot)$ on V corresponds to a unique $\mathcal{A} \in L(V, V^*)$ given by

$$a(x,y) = \mathcal{A}x(y) \ , \qquad x, y \in V \ . \tag{5.1}$$

Conversely, if $\mathcal{A} \in \mathcal{L}(V, V^*)$ is given, then Equation (5.1) defines a sesquilinear form on V.

Theorem 5.4 *Let V, p be a normed linear space and $a(\cdot,\cdot)$ a sesquilinear form on V. The following are equivalent:*

(a) *$a(\cdot,\cdot)$ is continuous at (θ,θ),*

(b) *$a(\cdot,\cdot)$ is continuous on $V \times V$,*

(c) *there is a constant $K \geq 0$ such that*

$$|a(x,y)| \leq Kp(x)p(y) \ , \qquad x, y \in V \ , \tag{5.2}$$

(d) *$\mathcal{A} \in \mathcal{L}(V, V')$.*

Proof: It is clear that (c) and (d) are equivalent, (c) implies (b), and (b) implies (a). We shall show that (a) implies (c). The continuity of $a(\cdot,\cdot)$ at (θ,θ) implies that there is a $\delta > 0$ such that $p(x) \leq \delta$ and $p(y) \leq \delta$ imply $|a(x,y)| \leq 1$. Thus, if $x \neq 0$ and $y \neq 0$ we obtain Equation (5.2) with $K = 1/\delta^2$.

When we consider real spaces (i.e., $\mathbb{K} = \mathbb{R}$) there is no distinction between linear and conjugate-linear functions. Then a sesquilinear form is linear in both variables and we call it *bilinear*.

6 Uniform Boundedness; Weak Compactness

A sequence $\{x_n\}$ in the Hilbert space H is called *weakly convergent* to $x \in H$ if $\lim_{n\to\infty}(x_n, v)_H = (x, v)_H$ for every $v \in H$. The weak limit x is clearly unique. Similarly, $\{x_n\}$ is *weakly bounded* if $|(x_n, v)_H|$ is bounded for every $v \in H$.

Our first result is a simple form of the *principle of uniform boundedness*.

Theorem 6.1 *A sequence $\{x_n\}$ is weakly bounded if and only if it is bounded.*

Proof: Let $\{x_n\}$ be weakly bounded. We first show that on some sphere, $s(x,r) = \{y \in H : \|y-x\| < r\}$, $\{x_n\}$ is uniformly bounded: there is a $K \geq 0$ with $|(x_n, y)_H| \leq K$ for all $y \in s(x,r)$. Suppose not. Then there is an integer n_1 and $y_1 \in s(0,1)$: $|(x_{n_1}, y_1)_H| > 1$. Since $y \mapsto (x_{n_1}, y)_H$ is continuous, there is an $r_1 < 1$ such that $|(x_{n_1}, y)_H| > 1$ for $y \in s(y_1, r_1)$. Similarly, there is an integer $n_2 > n_1$ and $\overline{s(y_2, r_2)} \subset s(y_1, r_1)$ such that $r_2 < 1/2$

and $|(x_{n_2}, y)_H| > 2$ for $y \in s(y_2, r_2)$. We inductively define $\overline{s(y_j, r_j)} \subset s(y_{j-1}, r_{j-1})$ with $r_j < 1/j$ and $|(x_{n_j}, y)_H| > j$ for $y \in s(y_j, r_j)$. Since $\|y_m - y_n\| < 1/n$ if $m > n$ and H is complete, $\{y_n\}$ converges to some $y \in H$. But then $y \in s(y_j, r_j)$, hence $|(x_{n_j}, y)_H| > j$ for all $j \geq 1$, a contradiction.

Thus $\{x_n\}$ is uniformly bounded on some sphere $s(y, r) : |(x_n, y+rz)_H| \leq K$ for all z with $\|z\| \leq 1$. If $\|z\| \leq 1$, then

$$|(x_n, z)_H| = (1/r)|x_n, y + rz)_H - (x_n, y)_H| \leq 2K/r \ ,$$

so $\|x_n\| \leq 2K/r$ for all n.

We next show that bounded sequences have weakly convergent subsequences.

Lemma *If $\{x_n\}$ is bounded in H and D is a dense subset of H, then $\lim_{n\to\infty}(x_n, v)_H = (x, v)_H$ for all $v \in D$ (if and) only if $\{x_n\}$ converges weakly to x.*

Proof: Let $\varepsilon > 0$ and $v \in H$. There is a $z \in D$ with $\|v - z\| < \varepsilon$ and we obtain

$$\begin{aligned} |(x_n - x, v)_H| &\leq |(x_n, v - z)_H| + |(z, x_n - x)_H| + |(x, v - z)_H| \\ &< \varepsilon\|x_n\| + |(z, x_n - x)_H| + \varepsilon\|x\| \ . \end{aligned}$$

Hence, for all n sufficiently large (depending on z), we have $|(x_n - x, v)_H| < 2\varepsilon \sup\{\|x_m\| : m \geq 1\}$. Since $\varepsilon > 0$ is arbitrary, the result follows.

Theorem 6.2 *Let the Hilbert space H have a countable dense subset $D = \{y_n\}$. If $\{x_n\}$ is a bounded sequence in H, then it has a weakly convergent subsequence.*

Proof: Since $\{(x_n, y_1)_H\}$ is bounded in $\mathbb{K}$, there is a subsequence $\{x_{1,n}\}$ of $\{x_n\}$ such that $\{(x_{1,n}, y_1)_H\}$ converges. Similarly, for each $j \geq 2$ there is a subsequence $\{x_{j,n}\}$ of $\{x_{j-1,n}\}$ such that $\{(x_{j,n}, y_k)_H\}$ converges in $\mathbb{K}$ for $1 \leq k \leq j$. It follows that $\{x_{n,n}\}$ is a subsequence of $\{x_n\}$ for which $\{(x_{n,n}, y_k)_H\}$ converges for every $k \geq 1$.

From the preceding remarks, it suffices to show that if $\{(x_n, y)_H\}$ converges in $\mathbb{K}$ for every $y \in D$, then $\{x_k\}$ has a weak limit. So, we define $f(y) = \lim_{n\to\infty}(x_n, y)_H$, $y \in \langle D \rangle$, where $\langle D \rangle$ is the subspace of all linear combinations of elements of D. Clearly f is linear; f is continuous, since $\{x_n\}$ is bounded, and has by Theorem 3.1 a unique extension $f \in H'$. But then there is by Theorem 4.5 an $x \in H$ such that $f(y) = (x, y)_H$, $y \in H$. The Lemma above shows that x is the weak limit of $\{x_n\}$.

Any seminormed space which has a countable and dense subset is called *separable*. Theorem 6.2 states that any bounded set in a separable Hilbert space is *relatively sequentially weakly compact*. This result holds in any reflexive Banach space, but all the function spaces which we shall consider are separable Hilbert spaces, so Theorem 6.2 will suffice for our needs.

7 Expansion in Eigenfunctions

7.1

We consider the Fourier series of a vector in the scalar product space H with respect to a given set of orthogonal vectors. The sequence $\{v_j\}$ of vectors in H is called *orthogonal* if $(v_i, v_j)_H = 0$ for each pair i, j with $i \neq j$. Let $\{v_j\}$ be such a sequence of non-zero vectors and let $u \in H$. For each j we define the *Fourier coefficient* of u with respect to v_j by $c_j = (u, v_j)_H / (v_j, v_j)_H$. For each $n \geq 1$ it follows that $\sum_{j=1}^{n} c_j v_j$ is the projection of u on the subspace M_n spanned by $\{v_1, v_2, \ldots, v_n\}$. This follows from Theorem 4.4 by noting that $u - \sum_{j=1}^{n} c_j v_j$ is orthogonal to each v_i, $1 \leq j \leq n$, hence belongs to $M_n^{\perp}$. We call the sequence of vectors *orthonormal* if they are orthogonal and if $(v_j, v_j)_H = 1$ for each $j \geq 1$.

Theorem 7.1 *Let $\{v_j\}$ be an orthonormal sequence in the scalar product space H and let $u \in H$. The Fourier coefficients of u are given by $c_j = (u, v_j)_H$ and satisfy*

$$\sum_{j=1}^{\infty} |c_j|^2 \leq \|u\|^2 \ . \tag{7.1}$$

Also we have $u = \sum_{j=1}^{\infty} c_j v_j$ if and only if equality holds in (7.1).

Proof: Let $u_n \equiv \sum_{j=1}^{n} c_j v_j$, $n \geq 1$. Then $u - u_n \perp u_n$ so we obtain

$$\|u\|^2 = \|u - u_n\|^2 + \|u_n\|^2 \ , \qquad n \geq 1 \ . \tag{7.2}$$

But $\|u_n\|^2 = \sum_{j=1}^{n} |c_j|^2$ follows since the set $\{v_i, \ldots, v_n\}$ is orthonormal, so we obtain $\sum_{j=1}^{n} |c_j|^2 \leq \|u\|^2$ for all n, hence (7.1) holds. It follows from (7.2) that $\lim_{n\to\infty} \|u - u_n\| - 0$ if and only if equality holds in (7.1).

The inequality (7.1) is *Bessel's inequality* and the corresponding equality is called *Parseval's equation*. The series $\sum_{j=1}^{\infty} c_j v_j$ above is the *Fourier series* of u with respect to the orthonormal sequence $\{v_j\}$.

Theorem 7.2 *Let $\{v_j\}$ be an orthonormal sequence in the scalar product space H. Then every element of H equals the sum of its Fourier series if and only if $\{v_j\}$ is a* basis *for H, that is, its linear span is dense in H.*

Proof: Suppose $\{v_j\}$ is a basis and let $u \in H$ be given. For any $\varepsilon > 0$, there is an $n \geq 1$ for which the linear span M of the set $\{v_1, v_2, \ldots, v_n\}$ contains an element which approximates u within ε. That is, $\inf\{\|u - w\| : w \in M\} < \varepsilon$. If u_n is given as in the proof of Theorem 7.1, then we have $u - u_n \in M^{\perp}$. Hence, for any $w \in M$ we have

$$\|u - u_n\|^2 = (u - u_n, u - w)_H \leq \|u - u_n\| \, \|u - w\| \ ,$$

since $u_n - w \in M$. Taking the infimum over all $w \in M$ then gives

$$\|u - u_n\| \leq \inf\{\|u - w\| : w \in M\} < \varepsilon \ . \tag{7.3}$$

Thus, $\lim_{n\to\infty} u_n = u$. The converse is clear.

7.2

Let $T \in \mathcal{L}(H)$. A non-zero vector $v \in H$ is called an *eigenvector* of T if $T(v) = \lambda v$ for some $\lambda \in \mathbb{K}$. The number λ is the *eigenvalue* of T corresponding to v. We shall show that certain operators possess a rich supply of eigenvectors. These eigenvectors form an orthonormal sequence to which we can apply the preceding Fourier series expansion techniques.

An operator $T \in \mathcal{L}(H)$ is called *self-adjoint* if $(Tu, v)_H = (u, Tv)_H$ for all $u, v \in H$. A self-adjoint T is called *non-negative* if $(Tu, u)_H \geq 0$ for all $u \in H$.

Lemma 7.3 *If $T \in \mathcal{L}(H)$ is non-negative self-adjoint, then $\|Tu\| \leq \|T\|^{1/2}(Tu, u)_H^{1/2}$, $u \in H$.*

Proof: The sesquilinear form $[u, v] \equiv (Tu, v)_H$ satisfies the first two scalar-product axioms and this is sufficient to obtain

$$|[u, v]|^2 \leq [u, u][v, v] \ , \qquad u, v \in H \ . \tag{7.4}$$

(If either factor on the right side is strictly positive, this follows from the proof of Theorem 4.1. Otherwise, $0 \leq [u+tv, u+tv] = 2t[u,v]$ for all $t \in \mathbb{R}$, hence, both sides of (7.4) are zero.) The desired result follows by setting $v = T(u)$ in (7.4).

The operators we shall consider are the compact operators. If V, W are seminormed spaces, then $T \in \mathcal{L}(V,W)$ is called *compact* if for any bounded sequence $\{u_n\}$ in V its image $\{Tu_n\}$ has a subsequence which converges in W. The essential fact we need is the following.

Lemma 7.4 *If $T \in \mathcal{L}(H)$ is self-adjoint and compact, then there exists a vector v with $\|v\| = 1$ and $T(v) = \mu v$, where $|\mu| = \|T\|_{\mathcal{L}(H)} > 0$.*

Proof: If λ is defined to be $\|T\|_{\mathcal{L}(H)}$, it follows from Theorem 2.5 that there is a sequence u_n in H with $\|u_n\| = 1$ and $\lim_{n\to\infty} \|Tu_n\| = \lambda$. Then $((\lambda^2 - T^2)u_n, u_n)_H = \lambda^2 - \|Tu_n\|^2$ converges to zero. The operator $\lambda^2 - T^2$ is non-negative self-adjoint so Lemma 7.3 implies $\{(\lambda^2 - T^2)u_n\}$ converges to zero. Since T is compact we may replace $\{u_n\}$ by an appropriate subsequence for which $\{Tu_n\}$ converges to some vector $w \in H$. Since T is continuous there follows $\lim_{n\to\infty}(\lambda^2 u_n) = \lim_{n\to\infty} T^2 u_n = Tw$, so $w = \lim_{n\to\infty} Tu_n = \lambda^{-2}T^2(w)$. Note that $\|w\| = \lambda$ and $T^2(w) = \lambda^2 w$. Thus, either $(\lambda + T)w \neq 0$ and we can choose $v = (\lambda + T)w/\|(\lambda + T)w\|$, or $(\lambda + T)w = 0$, and we can then choose $v = w/\|w\|$. Either way, the desired result follows.

Theorem 7.5 *Let H be a scalar product space and let $T \in \mathcal{L}(H)$ be self-adjoint and compact. Then there is an orthonormal sequence $\{v_j\}$ of eigenvectors of T for which the corresponding sequence of eigenvalues $\{\lambda_j\}$ converges to zero and the eigenvectors are a basis for* $\mathrm{Rg}(T)$.

Proof: By Lemma 7.4 it follows that there is a vector v_1 with $\|v_1\| = 1$ and $T(v_1) = \lambda_1 v_1$ with $|\lambda_1| = \|T\|_{\mathcal{L}(H)}$. Set $H_1 = \{v_1\}^\perp$ and note $T\{H_1\} \subset H_1$. Thus, the restriction $T|_{H_1}$ is self-adjoint and compact so Lemma 7.4 implies the existence of an eigenvector v_2 of T of unit length in H_1 with eigenvalue λ_2 satisfying $|\lambda_2| = \|T\|_{\mathcal{L}(H_1)} \leq |\lambda_1|$. Set $H_2 = \{v_1, v_2\}^\perp$ and continue this procedure to obtain an orthonormal sequence $\{v_j\}$ in H and sequence $\{\lambda_j\}$ in $\mathbb{R}$ such that $T(v_j) = \lambda_j v_j$ and $|\lambda_{j+1}| \leq |\lambda_j|$ for $j \geq 1$.

Suppose the sequence $\{\lambda_j\}$ is eventually zero; let n be the first integer for which $\lambda_n = 0$. Then $H_{n-1} \subset K(T)$, since $T(v_j) = 0$ for $j \geq n$. Also we see $v_j \in \mathrm{Rg}(T)$ for $j < n$, so $\mathrm{Rg}(T)^\perp \subset \{v_1, v_2, \ldots, v_{n-1}\}^\perp = H_{n-1}$ and from

Theorem 5.2 follows $K(T) = \mathrm{Rg}(T)^{\perp} \subset H_{n-1}$. Therefore $K(T) = H_{n-1}$ and $\mathrm{Rg}(T)$ equals the linear span of $\{v_1, v_2, \ldots, v_{n-1}\}$.

Consider hereafter the case where each λ_j is different from zero. We claim that $\lim_{j\to\infty}(\lambda_j) = 0$. Otherwise, since $|\lambda_j|$ is decreasing we would have all $|\lambda_j| \geq \varepsilon$ for some $\varepsilon > 0$. But then

$$\|T(v_i) - T(v_j)\|^2 = \|\lambda_i v_i - \lambda_j v_j\|^2 = \|\lambda_i v_i\|^2 + \|\lambda_j v_j\|^2 \geq 2\varepsilon^2$$

for all $i \neq j$, so $\{T(v_j)\}$ has no convergent subsequence, a contradiction. We shall show $\{v_j\}$ is a basis for $\mathrm{Rg}(T)$. Let $w \in \mathrm{Rg}(T)$ and $\sum b_j v_j$ the Fourier series of w. Then there is a $u \in H$ with $T(u) = w$ and we let $\sum c_j v_j$ be the Fourier series of u. The coefficients are related by

$$b_j = (w, v_j)_H = (Tu, v_j)_H = (u, Tv_j)_H = \lambda_j c_j \ ,$$

so there follows $T(c_j v_j) = b_j v_j$, hence,

$$w - \sum_{j=1}^{n} b_j v_j = T\left(u - \sum_{j=1}^{n} c_j v_j\right) \ , \qquad n \geq 1 \ . \tag{7.5}$$

Since T is bounded by $|\lambda_{n+1}|$ on H_n, and since $\|u - \sum_{j=1}^{n} c_j v_j\| \leq \|u\|$ by (7.2), we obtain from (7.5) the estimate

$$\left\| w - \sum_{j=1}^{n} b_j u_j \right\| \leq |\lambda_{n+1}| \cdot \|u\| \ , \qquad n \geq 1 \ . \tag{7.6}$$

Since $\lim_{j\to\infty} \lambda_j = 0$, we have $w = \sum_{j=1}^{\infty} b_j v_j$ as desired.

Exercises

1.1. Explain what "compatible" means in the Examples of Section 1.2.

1.2. Prove the Lemmas of Sections 1.3 and 1.4.

1.3. In Example (1.3.b), show V/M is isomorphic to $\mathbb{K}$.

1.4. Let $V = C(\bar{G})$ and $M = \{\varphi \in C(\bar{G}) : \varphi|_{\partial G} = 0\}$. Show V/M is isomorphic to $\{\varphi|_{\partial G} : \varphi \in C(\bar{G})\}$, the space of "boundary values" of functions in V.

1.5. In Example (1.3.c), show $\hat{\varphi}_1 = \hat{\varphi}_2$ if and only if φ_1 equals φ_2 on a neighborhood of ∂G. Find a space of functions isomorphic to V/M.

1.6. In Example (1.4.c), find $K(D)$ and $\mathrm{Rg}(D)$ when $V = \{\varphi \in C^1(\bar{G}) : \varphi(a) = \varphi(b)\}$.

1.7. Verify the last sentence in the Example of Section 1.5.

1.8. Let $M_\alpha \leq V$ for each $\alpha \in A$; show $\cap\{M_\alpha : \alpha \in A\} \leq V$.

2.1. Prove parts (d) and (e) of Lemma 2.1.

2.2. If V_1, p_1 and V_2, p_2 are seminormed spaces, show $p(x) \equiv p_1(x_1) + p_2(x_2)$ is a seminorm on the product $V_1 \times V_2$.

2.3. Let V, p be a seminormed space. Show limits are unique if and only if p is a norm.

2.4. Verify all Examples in Section 2.1.

2.5. Show $\cap_{\alpha\in A}\bar{S}_\alpha = \overline{\cap_{\alpha\in A}S_\alpha}$. Verify $\bar{S}$ = smallest closed set containing S.

2.6. Show $T : V, p \to W, q$ is continuous if and only if S closed in W, q implies $T(S)$ closed in V, p. If $T \in L(V, W)$, then T continuous if and only if $K(T)$ is closed.

2.7. The composition of continuous functions is continuous; $T \in \mathcal{L}(V, W)$, $S \in \mathcal{L}(U, V) \Rightarrow T \circ S \in \mathcal{L}(U, W)$ and $|T \circ S| \leq |T|\ |S|$.

2.8. Finish proof of Theorem 2.5.

2.9. Show V' is isomorphic to $\mathcal{L}(V, \mathbb{K})$; they are equal only if $\mathbb{K} = \mathbb{R}$.

3.1. Show that a closed subspace of a seminormed space is complete.

3.2. Show that a complete subspace of a normed space is closed.

3.3. Show that a Cauchy sequence is convergent if and only if it has a convergent subsequence.

3.4. Let V, p be a seminormed space and W, q a Banach space. Let the sequence $T_n \in \mathcal{L}(V, W)$ be given *uniformly bounded*: $|T_n|_{p,q} \leq K$ for all $n \geq 1$. Suppose that D is a dense subset of V and $\{T_n(x)\}$ converges in W for each $x \in D$. Then show $\{T_n(x)\}$ converges in W for each $x \in V$ and $T(x) = \lim T_n(x)$ defines $T \in \mathcal{L}(V, W)$. Show that completeness of W is necessary above.

3.5. Let V, p and W, q be as given above. Show $\mathcal{L}(V, W)$ is isomorphic to $\mathcal{L}(V/\operatorname{Ker}(p), W)$.

3.6. Prove the remark in Section 3.3 on uniqueness of a completion.

4.1. Show that the norms p_2 and r_2 of Section 2.1 are not obtained from scalar products.

4.2. Let M be a subspace of the scalar product space $V(\cdot,\cdot)$. Then the following are equivalent: M is dense in V, $M^\perp = \{\theta\}$, and $\|f\|_{V'} = \sup\{|(f, v)_V| : v \in M\}$ for every $f \in V'$.

4.3. Show $\lim x_n = x$ in V, $(\cdot,\cdot)$ if and only if $\lim \|x_n\| = \|x\|$ and $\lim f(x_n) = f(x)$ for all $f \in V'$.

4.4. If V is a scalar product space, show V' is a Hilbert space. Show that the Riesz map of V into V' is surjective only if V is complete.

5.1. Prove Theorem 5.2.

5.2. Prove Corollary 5.3.

5.3. Verify $T = i' \circ R \circ i$ in Section 5.3.

5.4. In the situation of Theorem 5.2, prove the following are equivalent: $\operatorname{Rg}(T)$ is closed, $\operatorname{Rg}(T^*)$ is closed, $\operatorname{Rg}(T) = K(T^*)^\perp$, and $\operatorname{Rg}(T^*) = K(T)^\perp$.

7.1. Let $G = (0,1)$ and $H = L^2(G)$. Show that the sequence $v_n(x) = 2\sin(n\pi x)$, $n \geq 1$ is orthonormal in H.

7.2. In Theorem 7.1, show that $\{u_n\}$ is a Cauchy sequence.

7.3. Show that the eigenvalues of a non-negative self-adjoint operator are all non-negative.

7.4. In the situation of Theorem 7.5, show $K(T)$ is the orthogonal complement of the linear span of $\{v_1, v_2, v_3, \ldots\}$.

Chapter II

Distributions and Sobolev Spaces

1 Distributions

1.1

We shall begin with some elementary results concerning the approximation of functions by very smooth functions. For each $\varepsilon > 0$, let $\varphi_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ be given with the properties

$$\varphi_\varepsilon \geq 0 \quad , \quad \operatorname{supp}(\varphi_\varepsilon) \subset \{x \in \mathbb{R}^n : |x| \leq \varepsilon\} \quad , \quad \int \varphi_\varepsilon = 1 \ .$$

Such functions are called *mollifiers* and can be constructed, for example, by taking an appropriate multiple of

$$\psi_\varepsilon(x) = \begin{cases} \exp(|x|^2 - \varepsilon^2)^{-1} \ , & |x| < \varepsilon \ , \\ 0 \ , & |x| \geq \varepsilon \ . \end{cases}$$

Let $f \in L^1(G)$, where G is open in $\mathbb{R}^n$, and suppose that the support of f satisfies $\operatorname{supp}(f) \subset\subset G$. Then the distance from $\operatorname{supp}(f)$ to ∂G is a positive number δ. We extend f as zero on the complement of G and denote the extension in $L^1(\mathbb{R}^n)$ also by f. Define for each $\varepsilon > 0$ the mollified function

$$f_\varepsilon(x) = \int_{\mathbb{R}^n} f(x-y)\varphi_\varepsilon(y)\, dy \ , \qquad x \in \mathbb{R}^n \ . \tag{1.1}$$

Lemma 1.1 *For each* $\varepsilon > 0$, $\operatorname{supp}(f_\varepsilon) \subset \operatorname{supp}(f) + \{y : |y| \leq \varepsilon\}$ *and* $f_\varepsilon \in C^\infty(\mathbb{R}^n)$.

Proof: The second result follows from Leibnitz' rule and the representation

$$f_\varepsilon(x) = \int f(s)\varphi_\varepsilon(x-s)\,ds\ .$$

The first follows from the observation that $f_\varepsilon(x) \neq 0$ only if $x \in \operatorname{supp}(f) + \{y : |y| \leq \varepsilon\}$. Since $\operatorname{supp}(f)$ is closed and $\{y : |y| \leq \varepsilon\}$ is compact, it follows that the indicated set sum is closed and, hence, contains $\operatorname{supp}(f_\varepsilon)$.

Lemma 1.2 *If $f \in C_0(G)$, then $f_\varepsilon \to f$ uniformly on G. If $f \in L^p(G)$, $1 \leq p < \infty$, then $\|f_\varepsilon\|_{L^p(G)} \leq \|f\|_{L^p(G)}$ and $f_\varepsilon \to f$ in $L^p(G)$.*

Proof: The first result follows from the estimate

$$\begin{aligned}|f_\varepsilon(x) - f(x)| &\leq \int |f(x-y) - f(x)|\varphi_\varepsilon(y)\,dy \\ &\leq \sup\{|f(x-y) - f(x)| : x \in \operatorname{supp}(f)\ ,\ |y| \leq \varepsilon\}\end{aligned}$$

and the uniform continuity of f on its support. For the case $p = 1$ we obtain

$$\|f_\varepsilon\|_{L^1(G)} \leq \int\!\!\int |f(x-y)|\varphi_\varepsilon(y)\,dy\,dx = \int \varphi_\varepsilon \cdot \int |f|$$

by Fubini's theorem, since $\int |f(x-y)|\,dx = \int |f|$ for each $y \in \mathbb{R}^n$ and this gives the desired estimate. If $p = 2$ we have for each $\psi \in C_0(G)$

$$\begin{aligned}\left|\int f_\varepsilon(x)\psi(x)\,dx\right| &\leq \int\!\!\int |f(x-y)\psi(x)|\,dx\ \varphi_\varepsilon(y)\,dy \\ &\leq \int \|f\|_{L^2(G)}\|\psi\|_{L^2(G)}\varphi_\varepsilon(y)\,dy = \|f\|_{L^2(G)}\|\psi\|_{L^2(G)}\end{aligned}$$

by computations similar to the above, and the result follows since $C_0(G)$ is dense in $L^2(G)$. (We shall not use the result for $p \neq 1$ or 2, but the corresponding result is proved as above but using the Hölder inequality in place of Cauchy-Schwarz.)

Finally we verify the claim of convergence in $L^p(G)$. If $\eta > 0$ we have a $g \in C_0(G)$ with $\|f - g\|_{L^p} \leq \eta/3$. The above shows $\|f_\varepsilon - g_\varepsilon\|_{L^p} \leq \eta/3$ and we obtain

$$\begin{aligned}\|f_\varepsilon - f\|_{L^p} &\leq \|f_\varepsilon - g_\varepsilon\|_{L^p} + \|g_\varepsilon - g\|_{L^p} + \|g - f\|_{L^p} \\ &\leq 2\eta/3 + \|g_\varepsilon - g\|_{L^p}\ .\end{aligned}$$

For ε sufficiently small, the support of $g_\varepsilon - g$ is bounded (uniformly) and $g_\varepsilon \to g$ uniformly, so the last term converges to zero as $\varepsilon \to 0$.

The preceding results imply the following.

Theorem 1.3 *$C_0^\infty(G)$ is dense in $L^p(G)$.*

Theorem 1.4 *For every $K \subset\subset G$ there is a $\varphi \in C_0^\infty(G)$ such that $0 \le \varphi(x) \le 1$, $x \in G$, and $\varphi(x) = 1$ for all x in some neighborhood of K.*

Proof: Let δ be the distance from K to ∂G and $0 < \varepsilon < \varepsilon + \varepsilon' < \delta$. Let $f(x) = 1$ if $\mathrm{dist}(x, K) \le \varepsilon'$ and $f(x) = 0$ otherwise. Then f_ε has its support within $\{x : \mathrm{dist}(x, K) \le \varepsilon + \varepsilon'\}$ and it equals 1 on $\{x : \mathrm{dist}(x, K) \le \varepsilon' - \varepsilon\}$, so the result follows if $\varepsilon < \varepsilon'$.

1.2

A *distribution* on G is defined to be a conjugate-linear functional on $C_0^\infty(G)$. That is, $C_0^\infty(G)^*$ is the linear space of distributions on G, and we also denote it by $\mathcal{D}^*(G)$.

Example. The space $L^1_{\mathrm{loc}}(G) = \cap\{L^1(K) : K \subset\subset G\}$ of locally integrable functions on G can be identified with a subspace of distributions on G as in the Example of I.1.5. That is, $f \in L^1_{\mathrm{loc}}(G)$ is assigned the distribution $T_f \in C_0^\infty(G)^*$ defined by

$$T_f(\varphi) = \int_G f\bar{\varphi} \ , \qquad \varphi \in C_0^\infty(G) \ , \tag{1.2}$$

where the Lebesgue integral (over the support of φ) is used. Theorem 1.3 shows that $T : L^1_{\mathrm{loc}}(G) \to C_0^\infty(G)^*$ is an injection. In particular, the (equivalence classes of) functions in either of $L^1(G)$ or $L^2(G)$ will be identified with a subspace of $\mathcal{D}^*(G)$.

1.3

We shall define the derivative of a distribution in such a way that it agrees with the usual notion of derivative on those distributions which arise from continuously differentiable functions. That is, we want to define $\partial^\alpha : \mathcal{D}^*(G) \to \mathcal{D}^*(G)$ so that

$$\partial^\alpha(T_f) = T_{D^\alpha f} \quad , \quad |\alpha| \le m \quad , \quad f \in C^m(G) \ .$$

But a computation with integration-by-parts gives

$$T_{D^\alpha f}(\varphi) = (-1)^{|\alpha|} T_f(D^\alpha \varphi) \ , \qquad \varphi \in C_0^\infty(G) \ ,$$

and this identity suggests the following.

Definition. The α^{th} partial derivative of the distribution T is the distribution $\partial^\alpha T$ defined by

$$\partial^\alpha T(\varphi) = (-1)^{|\alpha|} T(D^\alpha \varphi) \ , \qquad \varphi \in C_0^\infty(G) \ . \tag{1.3}$$

Since $D^\alpha \in L(C_0^\infty(G), C_0^\infty(G))$, it follows that $\partial^\alpha T$ is linear. Every distribution has derivatives of all orders and so also then does every function, e.g., in $L^1_{\text{loc}}(G)$, when it is identified as a distribution. Furthermore, by the very definition of the derivative ∂^α it is clear that ∂^α and D^α are compatible with the identification of $C^\infty(G)$ in $\mathcal{D}^*(G)$.

1.4

We give some examples of distributions on $\mathbb{R}$. Since we do not distinguish the function $f \in L^1_{\text{loc}}(\mathbb{R})$ from the functional T_f, we have the identity

$$f(\varphi) = \int_{-\infty}^{\infty} f(x)\overline{\varphi(x)}\, dx \ , \qquad \varphi \in C_0^\infty(\mathbb{R}) \ .$$

(a) If $f \in C^1(\mathbb{R})$, then

$$\partial f(\varphi) = -f(D\varphi) = -\int f(D\bar{\varphi}) = \int (Df)\bar{\varphi} = Df(\varphi) \ , \tag{1.4}$$

where the third equality follows by an integration-by-parts and all others are definitions. Thus, $\partial f = Df$, which is no surprise since the definition of derivative of distributions was rigged to make this so.

(b) Let the ramp and Heaviside functions be given respectively by

$$r(x) = \begin{cases} x \ , & x > 0 \\ 0 \ , & x \leq 0 \ , \end{cases} \qquad H(x) = \begin{cases} 1 \ , & x > 0 \\ 0 \ , & x < 0 \ . \end{cases}$$

Then we have

$$\partial r(\varphi) = -\int_0^\infty x D\bar{\varphi}(x)\, dx = \int_{-\infty}^{\infty} H(x)\bar{\varphi}(x)\, dx = H(\varphi) \ , \qquad \varphi \in C_0^\infty(G) \ ,$$

so we have $\partial r = H$, although $Dr(0)$ does not exist.

(c) The derivative of the non-continuous H is given by

$$\partial H(\varphi) = -\int_0^\infty D\bar{\varphi} = \bar{\varphi}(0) = \delta(\varphi) \ , \qquad \varphi \in C_0^\infty(G) \ ;$$

that is, $\partial H = \delta$, the Dirac functional. Also, it follows directly from the definition of derivative that

$$\partial^m \delta(\varphi) = (-1)^m \overline{(D^m \varphi)}(0) \ , \qquad m \geq 1 \ .$$

(d) Letting $A(x) = |x|$ and $I(x) = x$, $x \in \mathbb{R}$, we observe that $A = 2r - I$ and then from above obtain by linearity

$$\partial A = 2H - 1 \quad , \quad \partial^2 A = 2\delta \ . \tag{1.5}$$

Of course, these could be computed directly from definitions.

(e) For our final example, let $f : \mathbb{R} \to \mathbb{K}$ satisfy $f|_{\mathbb{R}^-} \in C^\infty(-\infty, 0]$, $f|_{\mathbb{R}^+} \in C^\infty[0, \infty)$, and denote the jump in the various derivatives at 0 by

$$\sigma_m(f) = D^m f(0^+) - D^m f(0^-) \ , \qquad m \geq 0 \ .$$

Then we obtain

$$\begin{aligned} \partial f(\varphi) &= -\int_0^\infty f\overline{(D\varphi)} - \int_{-\infty}^0 f\overline{(D\varphi)} \\ &= \int_0^\infty (Df)\bar{\varphi} + f(0^+)\overline{\varphi(0)} + \int_{-\infty}^0 (Df)\bar{\varphi} - f(0^-)\overline{\varphi(0)} \\ &= Df(\varphi) + \sigma_0(f)\delta(\varphi) \ , \qquad \varphi \in C_0^\infty(G) \ . \end{aligned} \tag{1.6}$$

That is, $\partial f = Df + \sigma_0(f)\delta$, and the derivatives of higher order can be computed from this formula, e.g.,

$$\begin{aligned} \partial^2 f &= D^2 f + \sigma_1(f)\delta + \sigma_0(f)\partial\delta \ , \\ \partial^3 f &= D^3 f + \sigma_2(f)\delta + \sigma_1(f)\partial\delta + \sigma_0(f)\partial^2\delta \ . \end{aligned}$$

For example, we have

$$\begin{aligned} \partial(H \cdot \sin) &= H \cdot \cos \ , \\ \partial(H \cdot \cos) &= -H \cdot \sin + \delta \ , \end{aligned}$$

so $H \cdot \sin$ is a solution (generalized) of the ordinary differential equation

$$(\partial^2 + 1)y = \delta \ .$$

1.5

Before discussing further the interplay between ∂ and D we remark that to claim a distribution T is "constant" on $\mathbb{R}$, means that there is a number $c \in \mathbb{K}$ such that $T = T_c$, i.e., T arises from the locally integrable function whose value everywhere is c:

$$T(\varphi) = c \int_{\mathbb{R}} \bar{\varphi} \ , \qquad \varphi \in C_0^\infty(\mathbb{R}) \ .$$

Hence a distribution is constant if and only if it depends only on the mean value of each φ. This observation is the key to the proof of our next result.

Theorem 1.5 (a) *If S is a distribution on $\mathbb{R}$, then there exists another distribution T such that $\partial T = S$.*

(b) *If T_1 and T_2 are distributions on $\mathbb{R}$ with $\partial T_1 = \partial T_2$, then $T_1 - T_2$ is constant.*

Proof: First note that $\partial T = S$ if and only if

$$T(\psi') = -S(\psi) \ , \qquad \psi \in C_0^\infty(\mathbb{R}) \ .$$

This suggests we consider $H = \{\psi' : \psi \in C_0^\infty(\mathbb{R})\}$. H is a subspace of $C_0^\infty(\mathbb{R})$. Furthermore, if $\zeta \in C_0^\infty(\mathbb{R})$, it follows that $\zeta \in H$ if and only if $\int \zeta = 0$. In that case we have $\zeta = \psi'$, where

$$\psi(x) = \int_{-\infty}^{x} \zeta \ , \qquad x \in \mathbb{R} \ .$$

Thus $H = \{\zeta \in C_0^\infty(\mathbb{R}) : \int \zeta = 0\}$ and this equality shows H is the kernel of the functional $\varphi \mapsto \int \varphi$ on $C_0^\infty(\mathbb{R})$. (This implies H is a hyperplane, but we shall prove this directly.)

Choose $\varphi_0 \in C_0^\infty(\mathbb{R})$ with mean value unity:

$$\int_{\mathbb{R}} \varphi_0 = 1 \ .$$

We shall show $C_0^\infty(\mathbb{R}) = H \oplus \mathbb{K} \cdot \varphi_0$, that is, each φ can be written in exactly one way as the sum of a $\zeta \in H$ and a constant multiple of φ_0. To check the uniqueness of such a sum, let $\zeta_1 + c_1\varphi_0 = \zeta_2 + c_2\varphi_0$ with the $\zeta_1, \zeta_2 \in H$. Integrating both sides gives $c_1 = c_2$ and, hence, $\zeta_1 = \zeta_2$. To verify the existence of such a representation, for each $\varphi \in C_0^\infty(G)$ choose $c = \int \varphi$ and define $\zeta = \varphi - c\varphi_0$. Then $\zeta \in H$ follows easily and we are done.

To finish the proof of (a), it suffices by our remark above to define T on H, for then we can extend it to all of $C_0^\infty(\mathbb{R})$ by linearity after choosing, e.g., $T\varphi_0 = 0$. But for $\zeta \in H$ we can define

$$T(\zeta) = -S(\psi) \ , \qquad \psi(x) = \int_{-\infty}^{x} \zeta \ ,$$

since $\psi \in C_0^\infty(\mathbb{R})$ when $\zeta \in H$.

Finally, (b) follows by linearity and the observation that $\partial T = 0$ if and only if T vanishes on H. But then we have

$$T(\varphi) = T(c\varphi_0 + \zeta) = T(\varphi_0)\bar{c} = T(\varphi_0) \int \bar{\varphi}$$

and this says T is the constant $T(\varphi_0) \in \mathbb{K}$.

Theorem 1.6 *If $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous, then $g = Df$ defines $g(x)$ for almost every $x \in \mathbb{R}$, $g \in L^1_{\text{loc}}(\mathbb{R})$, and $\partial f = g$ in $\mathcal{D}^*(\mathbb{R})$. Conversely, if T is a distribution on $\mathbb{R}$ with $\partial T \in L^1_{\text{loc}}(\mathbb{R})$, then $T (= T_f) = f$ for some absolutely continuous f, and $\partial T = Df$.*

Proof: With f and g as indicated, we have $f(x) = \int_0^x g + f(0)$. Then an integration by parts shows that

$$\int f(D\bar{\varphi}) = -\int g\bar{\varphi} \ , \qquad \varphi \in C_0^\infty(\mathbb{R}) \ ,$$

so $\partial f = g$. (This is a trivial extension of (1.4).) Conversely, let $g = \partial T \in L^1_{\text{loc}}(\mathbb{R})$ and define $h(x) = \int_0^x g$, $x \in \mathbb{R}$. Then h is absolutely continuous and from the above we have $\partial h = g$. But $\partial(T - h) = 0$, so Theorem 1.5 implies that $T = h + c$ for some constant $c \in \mathbb{K}$, and we have the desired result with $f(x) = h(x) + c$, $x \in \mathbb{R}$.

1.6

Finally, we give some examples of distributions on $\mathbb{R}^n$ and their derivatives.

(a) If $f \in C^m(\mathbb{R}^n)$ and $|\alpha| \le m$, we have

$$\partial^\alpha f(\varphi) = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f D^\alpha \bar{\varphi} = \int_{\mathbb{R}^n} D^\alpha f \cdot \bar{\varphi} = (D^\alpha f)(\varphi) \ , \qquad \varphi \in C_0^\infty(\mathbb{R}^n) \ .$$

(The first and last equalities follow from definitions, and the middle one is a computation.) Thus $\partial^\alpha f = D^\alpha f$ essentially because of our definition of ∂^α.

(b) Let

$$r(x) = \begin{cases} x_1 x_2 \dots x_n \ , & \text{if all } x_j \geq 0 \ , \\ 0 \ , & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \partial_1 r(\varphi) = -r(D_1\varphi) &= -\int_0^\infty \dots \int_0^\infty (x_1 \dots x_n) D_1 \varphi \, dx_1 \dots dx_n \\ &= \int_0^\infty \dots \int_0^\infty x_2 \dots x_n \ \overline{\varphi(x)} \, dx_1 \dots dx_n \ . \end{aligned}$$

Similarly,

$$\partial_2 \partial_1 r(\varphi) = \int_0^\infty \dots \int_0^\infty x_3 \dots x_n \ \overline{\varphi(x)} \, dx \ ,$$

and

$$\partial^{(1,1,\dots,1)} r(\varphi) = \int_{\mathbb{R}^n} H(x) \overline{\varphi(x)} \, dx = H(\varphi) \ ,$$

where H is the Heaviside function (= functional)

$$H(x) = \begin{cases} 1 \ , & \text{if all } x_j \geq 0 \ , \\ 0 \ , & \text{otherwise.} \end{cases}$$

(c) The derivatives of the Heaviside functional will appear as distributions given by integrals over subspaces of $\mathbb{R}^n$. In particular, we have

$$\partial_1 H(\varphi) = -\int_0^\infty \dots \int_0^\infty D_1 \overline{\varphi(x)} \, dx = \int_0^\infty \dots \int_0^\infty \bar{\varphi}(0, x_2, \dots, x_n) \, dx_2 \dots dx_n \ ,$$

a distribution whose value is determined by the restriction of φ to $\{0\} \times \mathbb{R}^{n-1}$,

$$\partial_2 \partial_1 H(\varphi) = \int_0^\infty \dots \int_0^\infty \bar{\varphi}(0, 0, x_3, \dots, x_n) \, dx_3 \dots dx_n \ ,$$

a distribution whose value is determined by the restriction of φ to $\{0\} \times \{0\} \times \mathbb{R}^{n-2}$, and, similarly,

$$\partial^{(1,1,\dots,1)} H(\varphi) = \overline{\varphi(0)} = \delta(\varphi) \ ,$$

where δ is the Dirac functional which evaluates at the origin.

(d) Let S be an $(n-1)$-dimensional C^1 manifold (cf. Section 2.3) in $\mathbb{R}^n$ and suppose $f \in C^\infty(\mathbb{R}^n \sim S)$ with f having at each point of S a limit from each side of S. For each j, $1 \leq j \leq n$, we denote by $\sigma_j(f)$ the jump in f at the surface S in the direction of increasing x_j. (Note that $\sigma_j(f)$ is then a function on S.) Then we have

$$\begin{aligned}\partial_1 f(\varphi) &= -f(D_1\varphi) = -\int_{\mathbb{R}^n} f(x)D_1\varphi(x)\,dx\\ &= \int_{\mathbb{R}^n}(D_1 f)\overline{(\varphi)}(x)\,dx + \int\cdots\int \sigma_1(f)\overline{\varphi(s)}\,dx_2\ldots dx_n\end{aligned}$$

where $s = s(x_2,\ldots,x_n)$ is the point on S which (locally) projects onto $(0,x_2,\ldots,x_n)$. Recall that a surface integral over S is given by

$$\int_S F\,ds = \int_A F\cdot\sec(\theta_1)\,dA$$

when S projects (injectively) onto a region A in $\{0\}\times\mathbb{R}^{n-1}$ and θ_1 is the angle between the x_1-axis and the unit normal ν to S. Thus we can write the above as

$$\partial_1 f(\varphi) = D_1 f(\varphi) + \int_S \sigma_1(f)\cos(\theta_1)\bar{\varphi}\,dS\ .$$

However, in this representation it is clear that the integral is independent of the direction in which S is crossed, since both $\sigma_1(f)$ and $\cos(\theta_1)$ change sign when the direction is reversed. We need only to check that $\sigma_1(f)$ is evaluated in the same direction as the normal $\nu = (\cos(\theta_1),\cos(\theta_2),\ldots,\cos(\theta_n))$. Finally, our assumption on f shows that $\sigma_1(f) = \sigma_2(f) = \cdots = \sigma_n(f)$, and we denote this common value by $\sigma(f)$ in the formulas

$$\partial_j f(\varphi) = (D_j f)(\varphi) + \int_S \sigma(f)\cos(\theta_j)\bar{\varphi}\,dS\ .$$

These generalize the formula (1.6).

(e) Suppose G is an open, bounded and connected set in $\mathbb{R}^n$ whose boundary ∂G is a C^1 manifold of dimension $n-1$. At each point $s\in\partial G$ there is a unit *normal vector* $\nu = (\nu_1,\nu_2,\ldots,\nu_n)$ whose components are direction cosines, i.e., $\nu_j = \cos(\theta_j)$, where θ_j is the angle between ν and the x_j axis. Suppose $f\in C^\infty(\bar{G})$ is given. Extend f to $\mathbb{R}^n$ by setting $f(x)=0$ for $x\notin\bar{G}$. In $C_0^\infty(\mathbb{R}^n)^*$ we have by Green's second identity (cf. Exercise 1.6)

$$\begin{aligned}\left(\sum_{j=1}^n \partial_j^2 f\right)(\varphi) &= \int_G f\left(\sum_{j=1}^n D_j^2\bar{\varphi}\right) = \int_G \sum_{j=1}^n (D_j^2 f)\bar{\varphi}\\ &\quad + \int_{\partial G}\left(f\frac{\partial\bar{\varphi}}{\partial\nu} - \bar{\varphi}\frac{\partial f}{\partial\nu}\right)dS\ ,\qquad \varphi\in C_0^\infty(\mathbb{R}^n)\ ,\end{aligned}$$

so the indicated distribution differs from the pointwise derivative by the functional

$$\varphi \mapsto \int_{\partial G} \left(f \frac{\partial \bar{\varphi}}{\partial \nu} - \bar{\varphi} \frac{\partial f}{\partial \nu} \right) dS \ ,$$

where $\frac{\partial f}{\partial \nu} = \nabla f \cdot \nu$ is the indicated (directional) *normal derivative* and $\nabla f = (\partial_1 f, \partial_2 f, \ldots, \partial_n f)$ denotes the *gradient* of f. Hereafter we shall also let

$$\Delta_n = \sum_{j=1}^{n} \partial_j^2$$

denote the *Laplace* differential operator in $\mathcal{D}^*(\mathbb{R}^n)$.

2 Sobolev Spaces

2.1

Let G be an open set in $\mathbb{R}^n$ and $m \geq 0$ an integer. Recall that $C^m(\bar{G})$ is the linear space of restrictions to $\bar{G}$ of functions in $C_0^m(\mathbb{R}^n)$. On $C^m(\bar{G})$ we define a scalar product by

$$(f, g)_{H^m(G)} = \sum \left\{ \int_G D^\alpha f \cdot \overline{D^\alpha g} : |\alpha| \leq m \right\}$$

and denote the corresponding norm by $\|f\|_{H^m(G)}$.

Define $H^m(G)$ to be the completion of the linear space $C^m(\bar{G})$ with the norm $\|\cdot\|_{H^m(G)}$. $H^m(G)$ is a Hilbert space which is important for much of our following work on boundary value problems. We note that the $H^0(G)$ norm and $L^2(G)$ norm coincide on $C(\bar{G})$, and that we have the inclusions

$$C_0(G) \subset C(\bar{G}) \subset L^2(G) \ .$$

Since we have identified $L^2(G)$ as the completion of $C_0(G)$ it follows that we must likewise identify $H^0(G)$ with $L^2(G)$. Thus $f \in H^0(G)$ if and only if there is a sequence $\{f_n\}$ in $C(\bar{G})$ (or $C_0(G)$) which is Cauchy in the $L^2(G)$ norm and $f_n \to f$ in that norm.

Let $m \geq 1$ and $f \in H^m(G)$. Then there is a sequence $\{f_n\}$ in $C^m(\bar{G})$ such that $f_n \to f$ in $H^m(G)$, hence $\{D^\alpha f_n\}$ is Cauchy in $L^2(G)$ for each multi-index α of order $\leq m$. For each such α, there is a unique $g_\alpha \in L^2(G)$ such that $D^\alpha f_n \to g_\alpha$ in $L^2(G)$. As indicated above, f is the limit of f_n, so, $f = g_\theta$, $\theta = (0, 0, \ldots, 0) \in \mathbb{R}^n$. Furthermore, if $|\alpha| \le m$ we have from an integration-by-parts

$$(D^\alpha f_n, \varphi)_{L^2(G)} = (-1)^{|\alpha|}(f_n, D^\alpha \varphi)_{L^2(G)} , \qquad \varphi \in C_0^\infty(G) .$$

Taking the limit as $n \to \infty$, we obtain

$$(g_\alpha, \varphi)_{L^2(G)} = (-1)^{|\alpha|}(f, D^\alpha \varphi)_{L^2(G)} , \qquad \varphi \in C_0^\infty(G) ,$$

so $g_\alpha = \partial^\alpha f$. That is, each $g_\alpha \in L^2(G)$ is uniquely determined as the α^{th} partial derivative of f in the sense of distribution on G. These remarks prove the following characterization.

Theorem 2.1 *Let G be open in $\mathbb{R}^n$ and $m \ge 0$. Then $f \in H^m(G)$ if and only if there is a sequence $\{f_n\}$ in $C^m(\bar{G})$ such that, for each α with $|\alpha| \le m$, the sequence $\{D^\alpha f_n\}$ is $L^2(G)$-Cauchy and $f_n \to f$ in $L^2(G)$. In that case we have $D^\alpha f_n \to \partial^\alpha f$ in $L^2(G)$.*

Corollary *$H^m(G) \subset H^k(G) \subset L^2(G)$ when $m \ge k \ge 0$, and if $f \in H^m(G)$ then $\partial^\alpha f \in L^2(G)$ for all α with $|\alpha| \le m$.*

We shall later find that $f \in H^m(G)$ if $\partial^\alpha f \in L^2(G)$ for all α with $|\alpha| \le m$ (cf. Section 5.1).

2.2

We define $H_0^m(G)$ to be the closure in $H^m(G)$ of $C_0^\infty(G)$. Generally, $H_0^m(G)$ is a proper subspace of $H^m(G)$. Note that for any $f \in H^m(G)$ we have

$$(\partial^\alpha f, \varphi)_{L^2(G)} = (-1)^{|\alpha|}(f, D^\alpha \varphi)_{L^2(G)} , \qquad |\alpha| \le m \ , \ \varphi \in C_0^\infty(G) .$$

We can extend this result by continuity to obtain the generalized integration-by-parts formula

$$(\partial^\alpha f, g)_{L^2(G)} = (-1)^{|\alpha|}(f, \partial^\alpha g)_{L^2(G)} , \quad f \in H^m(G) \ , \ g \in H_0^m(G) \ , \ |\alpha| \le m .$$

This formula suggests that $H_0^m(G)$ consists of functions in $H^m(G)$ which vanish on ∂G together with their derivatives through order $m-1$. We shall make this precise in the following (cf. Theorem 3.4).

Since $C_0^\infty(G)$ is dense in $H_0^m(G)$, each element of $H_0^m(G)'$ determines (by restriction to $C_0^\infty(G)$) a distribution on G and this correspondence is an injection. Thus we can identify $H_0^m(G)'$ with a space of distributions on G, and those distributions are characterized as follows.

Theorem 2.2 *$H_0^m(G)'$ is (identified with) the space of distributions on G which are the linear span of the set*

$$\{\partial^\alpha f : |\alpha| \le m\ ,\ f \in L^2(G)\}\ .$$

Proof: If $f \in L^2(G)$ and $|\alpha| \le m$, then

$$|\partial^\alpha f(\varphi)| \le \|f\|_{L^2(G)} \|\varphi\|_{H_0^m(G)}\ , \qquad \varphi \in C_0^\infty(G)\ ,$$

so $\partial^\alpha f$ has a (unique) continuous extension to $H_0^m(G)$. Conversely, if $T \in H_0^m(G)'$, there is an $h \in H_0^m(G)$ such that

$$T(\varphi) = (h, \varphi)_{H^m(G)}\ , \qquad \varphi \in C_0^\infty(G)\ .$$

But this implies $T = \sum_{|\alpha| \le m} (-1)^{|\alpha|} \partial^\alpha(\partial^\alpha h)$ and, hence, the desired result, since each $\partial^\alpha h \in L^2(G)$.

We shall have occasion to use the two following results, each of which suggests further that $H_0^m(G)$ is distinguished from $H^m(G)$ by boundary values.

Theorem 2.3 $H_0^m(\mathbb{R}^n) = H^m(\mathbb{R}^n)$. *(Note that the boundary of $\mathbb{R}^n$ is empty.)*

Proof: Let $\tau \in C_0^\infty(\mathbb{R}^n)$ with $\tau(x) = 1$ when $|x| \le 1$, $\tau(x) = 0$ when $|x| \ge 2$, and $0 \le \tau(x) \le 1$ for all $x \in \mathbb{R}^n$. For each integer $k \ge 1$, define $\tau_k(x) = \tau(x/k)$, $x \in \mathbb{R}^n$. Then for any $u \in H^m(\mathbb{R}^n)$ we have $\tau_k \cdot u \in H^m(\mathbb{R}^n)$ and (exercise) $\tau_k \cdot u \to u$ in $H^m(\mathbb{R}^n)$ as $k \to \infty$. Thus we may assume u has compact support. Letting G denote a sphere in $\mathbb{R}^n$ which contains the support of u, we have from Lemma 1.2 of Section 1.1 that the mollified functions $u_\varepsilon \to u$ in $L^2(G)$ and that $(D^\alpha u)_\varepsilon = D^\alpha(u_\varepsilon) \to \partial^\alpha u$ in $L^2(G)$ for each α with $|\alpha| \le m$. That is, $u_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ and $u_\varepsilon \to u$ in $H^m(\mathbb{R}^n)$.

Theorem 2.4 *Suppose G is an open set in $\mathbb{R}^n$ with* $\sup\{|x_1| : (x_1, x_2, \ldots, x_n) \in G\} = K < \infty$. *Then*

$$\|\varphi\|_{L^2(G)} \le 2K \|\partial_1 \varphi\|_{L^2(G)}\ , \qquad \varphi \in H_0^1(G)\ .$$

Proof: We may assume $\varphi \in C_0^\infty(G)$, since this set is dense in $H_0^1(G)$. Then integrating the identity

$$D_1(x_1 \cdot |\varphi(x)|^2) = |\varphi(x)|^2 + x_1 \cdot D_1(|\varphi(x)|^2)$$

over G by the divergence theorem gives

$$\int_G |\varphi(x)|^2 = -\int_G x_1(D_1\varphi(x) \cdot \bar{\varphi}(x) + \varphi(x) \cdot D_1\bar{\varphi}(x))\,dx\ .$$

The right side is bounded by $2K\|D_1\varphi\|_{L^2(G)}\|\varphi\|_{L^2(G)}$, and this gives the result.

2.3

We describe a technique by which certain properties of $H^m(G)$ can be deduced from the corresponding property for $H_0^m(G)$ or $H^m(\mathbb{R}_+^n)$, where $\mathbb{R}_+^n = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > 0\}$ has a considerably simpler boundary. This technique is appropriate when, e.g., G is open and bounded in $\mathbb{R}^n$ and lies (locally) on one side of its boundary ∂G which we assume is a C^m-*manifold* of dimension $n-1$. Letting $Q = \{y \in \mathbb{R}^n : |y_j| \le 1,\ 1 \le j \le n\}$, $Q_0 = \{y \in Q : y_n = 0\}$, and $Q_+ = \{y \in Q : y_n > 0\}$, we can formulate this last condition as follows:

There is a collection $\{G_j : 1 \le j \le N\}$ of open bounded sets in $\mathbb{R}^n$ for which $\partial G \subset \cup\{G_j : 1 \le j \le N\}$ and a corresponding collection of functions $\varphi_j \in C^m(Q, G_j)$ with positive Jacobian $J(\varphi_j)$, $1 \le j \le N$, and φ_j is a bijection of Q, Q_+ and Q_0 onto G_j, $G_j \cap G$, and $G_j \cap \partial G$, respectively. For each j, the pair (φ_j, G_j) is a *coordinate patch* for the boundary.

Given the collection $\{(\varphi_j, G_j) : 1 \le j \le N\}$ of coordinate patches as above, we construct a corresponding collection of open sets F_j in $\mathbb{R}^n$ for which each $\bar{F}_j \subset G_j$ and $\cup\{F_j : 1 \le j \le N\} \supset \partial G$. Define $G_0 = G$ and $F_0 = G \sim \cup\{\bar{F}_j : 1 \le j \le N\}$, so $\bar{F}_0 \subset G_0$. Note also that $\bar{G} \subset G \cup \bigcup\{F_j : 1 \le j \le N\}$ and $G \subset \cup\{\bar{F}_j : 0 \le j \le N\}$. For each j, $0 \le j \le N$, let $\alpha_j \in C_0^\infty(\mathbb{R}^n)$ be chosen so that $0 \le \alpha_j(x) \le 1$ for all $x \in \mathbb{R}^n$, $\operatorname{supp}(\alpha_j) \subset G_j$, and $\alpha_j(x) = 1$ for $x \in \bar{F}_j$. Let $\alpha \in C_0^\infty(\mathbb{R}^n)$ be chosen with $0 \le \alpha(x) \le 1$ for all $x \in \mathbb{R}^n$, $\operatorname{supp}(\alpha) \subset G \cup \bigcup\{F_j : 1 \le j \le N\}$, and $\alpha(x) = 1$ for $x \in \bar{G}$. Finally, for each j, $0 \le j \le N$, we define $\beta_j(x) = \alpha_j(x)\alpha(x)/\sum_{k=0}^N \alpha_k(x)$ for $x \in \cup\{\bar{F}_j : 0 \le j \le N\}$ and $\beta_j(x) = 0$ for $x \in \mathbb{R}^n \sim \cup\{\bar{F}_j : 1 \le j \le N\}$. Then we have $\beta_j \in C_0^\infty(\mathbb{R}^n)$, β_j has support in G_j, $\beta_j(x) \ge 0$, $x \in \mathbb{R}^n$ and

$\sum\{\beta_j(x) : 0 \le j \le N\} = 1$ for each $x \in \bar{G}$. That is, $\{\beta_j : 0 \le j \le N\}$ is a *partition-of-unity* subordinate to the open cover $\{G_j : 0 \le j \le N\}$ of $\bar{G}$ and $\{\beta_j : 1 \le j \le N\}$ is a partition-of-unity subordinate to the open cover $\{G_j : 1 \le j \le N\}$ of ∂G.

Suppose we are given a $u \in H^m(G)$. Then we have $u = \sum_{j=0}^{N}\{\beta_j u\}$ on G and we can show that each pointwise product $\beta_j u$ is in $H^m(G \cap G_j)$ with support in G_j. This defines a function $H^m(G) \to H_0^m(G) \times \prod\{H^m(G \cap G_j) : 1 \le j \le N\}$, where $u \mapsto (\beta_0 u, \beta_1 u, \ldots, \beta_N u)$. This function is clearly linear, and from $\sum \beta_j = 1$ it follows that it is an injection. Also, since each $\beta_j u$ belongs to $H^m(G \cap G_j)$ with support in G_j for each $1 \le j \le N$, it follows that the composite function $(\beta_j u) \circ \varphi_j$ belongs to $H^m(Q^+)$ with support in Q. Thus, we have defined a linear injection

$$\begin{aligned} \Lambda : H^m(G) &\longrightarrow H_0^m(G) \times [H^m(Q^+)]^N \ , \\ u &\longmapsto (\beta_0 u, (\beta_1 u) \circ \varphi_1, \ldots, (\beta_N u) \circ \varphi_N) \ . \end{aligned}$$

Moreover, we can show that the product norm on Λu is equivalent to the norm of u in $H^m(G)$, so Λ is a continuous linear injection of $H^m(G)$ onto a closed subspace of the indicated product, and its inverse in continuous.

In a similar manner we can localize the discussion of functions on the boundary. In particular, $C^m(\partial G)$, the space of m times continuously differentiable functions on ∂G, is the set of all functions $f : \partial G \to \mathbb{R}$ such that $(\beta_j f) \circ \varphi_j \in C^m(Q_0)$ for each j, $1 \le j \le N$. The manifold ∂G has an intrinsic measure denoted by "ds" for which integrals are given by

$$\int_{\partial G} f\, ds = \sum_{j=1}^{N} \int_{\partial G \cap G_j} (\beta_j f)\, ds = \sum_{j=1}^{N} \int_{Q_0} (\beta_j f) \circ \varphi_j(y') J(\varphi_j)\, dy' \ ,$$

where $J(\varphi_j)$ is the indicated Jacobian and dy' denotes the usual (Lebesgue) measure on $Q_0 \subset \mathbb{R}^{n-1}$. Thus, we obtain a norm on $C(\partial G) = C^0(\partial G)$ given by $\|f\|_{L^2(\partial G)} = (\int_{\partial G} |f|^2\, ds)^{1/2}$, and the completion is the Hilbert space $L^2(\partial G)$ with the obvious scalar-product. We have a linear injection

$$\begin{aligned} \lambda : L^2(\partial G) &\longrightarrow [L^2(Q_0)]^N \\ f &\longmapsto ((\beta_1 f) \circ \varphi_1, \ldots, (\beta_N f) \circ \varphi_N) \end{aligned}$$

onto a closed subspace of the product, and both λ and its inverse are continuous.

3 Trace

We shall describe the sense in which functions in $H^m(G)$ have "boundary values" on ∂G when $m \geq 1$. Note that this is impossible in $L^2(G)$ since ∂G is a set of measure zero in $\mathbb{R}^n$. First, we consider the situation where G is the half-space $\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n) : x_n > 0\}$, for then $\partial G = \{(x', 0) : x' \in \mathbb{R}^{n-1}\}$ is the simplest possible (without being trivial). Also, the general case can be localized as in Section 2.3 to this case, and we shall use this in our final discussion of this section.

3.1

We shall define the (first) trace operator γ_0 when $G = \mathbb{R}^n_+ = \{x = (x', x_n) : x' \in \mathbb{R}^{n-1}, x_n > 0\}$, where we let x' denote the $(n-1)$-tuple $(x_1, x_2, \ldots, x_{n-1})$. For any $\varphi \in C^1(\bar{G})$ and $x' \in \mathbb{R}^{n-1}$ we have

$$|\varphi(x', 0)|^2 = -\int_0^\infty D_n(|\varphi(x', x_n)|^2)\, dx_n \ .$$

Integrating this identity over $\mathbb{R}^{n-1}$ gives

$$\begin{aligned} \|\varphi(\cdot, 0)\|^2_{L^2(\mathbb{R}^{n-1})} &\leq \int_{\mathbb{R}^n_+} [(D_n\varphi \cdot \bar{\varphi} + \varphi \cdot D_n\bar{\varphi}_n\,)]\, dx \\ &\leq 2\|D_n\varphi\|_{L^2(\mathbb{R}^n_+)} \|\varphi\|_{L^2(\mathbb{R}^n_+)} \ . \end{aligned}$$

The inequality $2ab \leq a^2 + b^2$ then gives us the estimate

$$\|\varphi(\cdot, 0)\|^2_{L^2(\mathbb{R}^{n-1})} \leq \|\varphi\|^2_{L^2(\mathbb{R}^n_+)} + \|D_n\varphi\|^2_{L^2(\mathbb{R}^n_+)} \ .$$

Since $C^1\overline{(\mathbb{R}^n_+)}$ is dense in $H^1(\mathbb{R}^n_+)$, we have proved the essential part of the following result.

Theorem 3.1 *The* trace *function* $\gamma_0 : C^1(\bar{G}) \to C^0(\partial G)$ *defined by*

$$\gamma_0(\varphi)(x') = \varphi(x', 0) \ , \qquad \varphi \in C^1(\bar{G}) \ , \ x' \in \partial G \ ,$$

(where $G = \mathbb{R}^n_+$*) has a unique extension to a continuous linear operator* $\gamma_0 \in \mathcal{L}(H^1(G), L^2(\partial G))$ *whose range is dense in* $L^2(\partial G)$*, and it satisfies*

$$\gamma_0(\beta \cdot u) = \gamma_0(\beta) \cdot \gamma_0(u) \ , \qquad \beta \in C^1(\bar{G}) \ , \ u \in H^1(G) \ .$$

Proof: The first part follows from the preceding inequality and Theorem I.3.1. If $\psi \in C_0^\infty(\mathbb{R}^{n-1})$ and τ is the truncation function defined in the proof of Theorem 2.3, then

$$\varphi(x) = \psi(x')\tau(x_n)\ , \qquad x = (x', x_n) \in \mathbb{R}^n_+$$

defines $\varphi \in C^1(\bar{G})$ and $\gamma_0(\varphi) = \psi$. Thus the range of γ_0 contains $C_0^\infty(\mathbb{R}^{n-1})$. The last identity follows by the continuity of γ_0 and the observation that it holds for $u \in C^1(\bar{G})$.

Theorem 3.2 *Let $u \in H^1(\mathbb{R}^n_+)$. Then $u \in H_0^1(\mathbb{R}^n_+)$ if and only if $\gamma_0(u) = 0$.*

Proof: If $\{u_n\}$ is a sequence in $C_0^\infty(\mathbb{R}^n_+)$ converging to u in $H^1(\mathbb{R}^n_+)$, then $\gamma_0(u) = \lim \gamma_0(u_n) = 0$ by Theorem 3.1.

Let $u \in H^1(\mathbb{R}^n_+)$ with $\gamma_0 u = 0$. If $\{\tau_j : j \geq 1\}$ denotes the sequence of truncating functions defined in the proof of Theorem 2.3, then $\tau_j u \to u$ in $H^1(\mathbb{R}^n_+)$ and we have $\gamma_0(\tau_j u) = \gamma_0(\tau_j)\gamma_0(u) = 0$, so we may assume that u has compact support in $\mathbb{R}^n$.

Let $\theta_j \in C^1(\mathbb{R}_+)$ be chosen such that $\theta_j(s) = 0$ if $0 < s \leq 1/j$, $\theta_j(s) = 1$ if $s \geq 2/j$, and $0 \leq \theta_j'(s) \leq 2j$ if $(1/j) \leq s \leq (2/j)$. Then the extension of $x \mapsto \theta_j(x_n)u(x', x_n)$ to all of $\mathbb{R}^n$ as 0 on $\mathbb{R}^n_-$ is a function in $H^1(\mathbb{R}^n)$ with support in $\{x : x_n \geq 1/j\}$, and (the proof of) Theorem 2.3 shows we may approximate such a function from $C_0^\infty(\mathbb{R}^n_+)$. Hence, we need only to show that $\theta_j u \to u$ in $H^1(\mathbb{R}^n_+)$.

It is an easy consequence of the Lebesgue dominated convergence theorem that $\theta_j u \to u$ in $L^2(\mathbb{R}^n_+)$ and for each k, $1 \leq k \leq n-1$, that $\partial_k(\theta_j u) = \theta_j(\partial_k u) \to \partial_k u$ in $L^2(\mathbb{R}^n_+)$ as $j \to \infty$. Similarly, $\theta_j(\partial_n u) \to \partial_n u$ and we have $\partial_n(\theta_j u) = \theta_j(\partial_n u) + \theta_j' u$, so we need only to show that $\theta_j' u \to 0$ in $L^2(\mathbb{R}^n_+)$ as $j \to \infty$.

Since $\gamma_0(u) = 0$ we have $u(x', s) = \int_0^s \partial_n u(x', t)\,dt$ for $x' \in \mathbb{R}^{n-1}$ and $s \geq 0$. From this follows the estimate

$$|u(x', s)|^2 \leq s \int_0^s |\partial_n u(x', t)|^2\,dt\ .$$

Thus, we obtain for each $x' \in \mathbb{R}^{n-1}$

$$\begin{aligned} \int_0^\infty |\theta_j'(s)u(x', s)|^2\,ds &\leq \int_0^{2/j} (2j)^2 s \int_0^s |\partial_n u(x', t)|^2\,dt\,ds \\ &\leq 8j \int_0^{2/j} \int_0^s |\partial_n u(x', t)|^2\,dt\,ds\ . \end{aligned}$$

Interchanging the order of integration gives

$$\begin{aligned}\int_0^\infty |\theta_j'(s)u(x',s)|^2\,ds &\le 8j\int_0^{2/j}\int_t^{2/j} |\partial_n u(x',t)|^2\,ds\,dt\\ &\le 16\int_0^{2/j} |\partial_n u(x',t)|^2\,dt\ .\end{aligned}$$

Integration of this inequality over $\mathbb{R}^{n-1}$ gives us

$$\|\theta_j' u\|^2_{L^2(\mathbb{R}^n_+)} \le 16\int_{\mathbb{R}^{n-1}\times[0,2/j]} |\partial_n u|^2\,dx$$

and this last term converges to zero as $j\to\infty$ since $\partial_n u$ is square-summable.

3.2

We can extend the preceding results to the case where G is a sufficiently smooth region in $\mathbb{R}^n$. Suppose G is given as in Section 2.3 and denote by $\{G_j : 0\le j\le N\}$, $\{\varphi_j : 1\le j\le N\}$, and $\{\beta_j : 0\le j\le N\}$ the open cover, corresponding local maps, and the partition-of-unity, respectively. Recalling the linear injections Λ and λ constructed in Section 2.3, we are led to consider function $\gamma_0 : H^1(G)\to L^2(\partial G)$ defined by

$$\begin{aligned}\gamma_0(u) &= \sum_{j=1}^N \Big(\gamma_0((\beta_j u)\circ\varphi_j)\Big)\circ\varphi_j^{-1}\\ &= \sum_{j=1}^N \gamma_0(\beta_j)\cdot(\gamma_0(u\circ\varphi_j)\varphi_j^{-1})\end{aligned}$$

where the equality follows from Theorem 3.1. This formula is precisely what is necessary in order that the following diagram commutes.

$$\begin{array}{ccccc} H^1(G) & \xrightarrow{\Lambda} & H^1_0(G)\times H^1(Q^+) & \times\cdots\times & H^1(Q^+)\\ \big\downarrow{\scriptstyle\gamma_0} & & \big\downarrow & & \big\downarrow{\scriptstyle\gamma_0}\\ L^2(\partial G) & \xrightarrow{\lambda} & L^2(Q^0) & \times\cdots\times & L^2(Q^0)\end{array}$$

Also, if $u\in C^1(\bar G)$ we see that $\gamma_0(u)$ is the restriction of u to ∂G. These remarks and Theorems 3.1 and 3.2 provide a proof of the following result.

Theorem 3.3 *Let G be a bounded open set in $\mathbb{R}^n$ which lies on one side of its boundary, ∂G, which we assume is a C^1-manifold. Then there exists a unique continuous and linear function $\gamma_0 : H^1(G) \to L^2(\partial G)$ such that for each $u \in C^1(\bar{G})$, $\gamma_0(u)$ is the restriction of u to ∂G. The kernel of γ_0 is $H_0^1(G)$ and its range is dense in $L^2(\partial G)$.*

This result is a special case of the trace theorem which we briefly discuss. For a function $u \in C^m(\bar{G})$ we define the various traces of normal derivatives given by

$$\gamma_j(u) = \left.\frac{\partial^j u}{\partial \nu^j}\right|_{\partial G} , \qquad 0 \le j \le m-1 .$$

Here ν denotes the unit outward normal on the boundary of G. When $G = \mathbb{R}_+^n$ (or G is localized as above), these are given by $\partial u/\partial \nu = -\partial_n u|_{x_n=0}$. Each γ_j can be extended by continuity to all of $H^m(G)$ and we obtain the following.

Theorem 3.4 *Let G be an open bounded set in $\mathbb{R}^n$ which lies on one side of its boundary, ∂G, which we assume is a C^m-manifold. Then there is a unique continuous linear function γ from $H^m(G)$ into $\prod_{j=0}^{m-1} H^{m-1-j}(\partial G)$ such that*

$$\gamma(u) = (\gamma_0 u, \gamma_1 u, \ldots, \gamma_{m-1}(u)) , \qquad u \in C^m(\bar{G}) .$$

The kernel of γ is $H_0^m(G)$ and its range is dense in the indicated product.

The Sobolev spaces over ∂G which appear in Theorem 3.4 can be defined locally. The range of the trace operator can be characterized by Sobolev spaces of fractional order and then one obtains a space of boundary values which is isomorphic to the quotient space $H^m(G)/H_0^m(G)$. Such characterizations are particularly useful when considering non-homogeneous boundary value problems and certain applications, but the preceding results will be sufficient for our needs.

4 Sobolev's Lemma and Imbedding

We obtained the spaces $H^m(G)$ by completing a class of functions with continuous derivatives. Our objective here is to show that each element of $H^m(G)$ is (represented by) a function with continuous derivatives up to a certain order which depends on m.

Let G be bounded and open in $\mathbb{R}^n$. We say G satisfies a *cone condition* if there is a $\rho > 0$ and $\gamma > 0$ such that each point $y \in \bar{G}$ is the vertex of a cone $K(y)$ of radius ρ and volume $\gamma\rho^n$ with $K(y) \subset \bar{G}$. Thus, γ is a measure of the angle of the cone. To be precise, a ball of radius ρ has volume $\omega_n \rho^n / n$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$, and the *angle* of the cone $K(y)$ is the ratio of these volumes given by $\gamma n/\omega_n$.

We shall derive an estimate on the value of a smooth function at a point $y \in \bar{G}$ in terms of the norm of $H^m(G)$ for some $m \geq 0$. Let $g \in C_0^\infty(\mathbb{R})$ satisfy $g \geq 0$, $g(t) = 1$ for $|t| \leq 1/2$, and $g(t) = 0$ for $|t| \geq 1$. Define $\tau(t) = g(t/\rho)$ and note that there are constants $A_k > 0$ such that

$$\left|\frac{d^k}{dt^k}\,\tau(t)\right| \leq \frac{A_k}{\rho^k}\ , \qquad \rho > 0\ . \tag{4.1}$$

Let $u \in C^m(\bar{G})$ and assume $2m > n$. If $y \in \bar{G}$ and $K(y)$ is the indicated cone, we integrate along these points $x \in K(y)$ on a given ray from the vertex y and obtain

$$\int_0^\rho D_r(\tau(r)u(x))\,dr = -u(y)\ ,$$

where $r = |x - y|$ for each such x. Thus, we obtain an integral over $K(y)$ in spherical coordinates given by

$$\int_\Omega \int_0^\rho D_r(\tau(r)u(x))\,dr\,d\omega = -u(y)\int_\Omega d\omega = -u(y)\gamma n/\omega_n$$

where ω is spherical angle and $\Omega = \gamma n/\omega_n$ is the total angle of the cone $K(y)$. We integrate by parts $m-1$ times and thereby obtain

$$u(y) = \frac{(-1)^m \omega_n}{\gamma n (m-1)!}\int_\Omega \int_0^\rho D_r^m(\tau u) r^{m-1}\,dr\,d\omega\ .$$

Changing this to Euclidean coordinates with volume element $dx = r^{n-1}\,dr\,d\omega$ gives

$$|u(y)| = \frac{\omega_n}{\gamma n(m-1)!}\int_{K(y)} D_r^m(\tau u) r^{m-n}\,dx\ .$$

The Cauchy-Schwartz inequality gives the estimate

$$|u(y)|^2 \leq \left(\frac{\omega_n}{\gamma n(m-1)!}\right)^2 \int_{K(y)} |D_r^m(\tau u)|^2\,dx\ \int_{K(y)} r^{2(m-n)}\,dx\ ,$$

and we use spherical coordinates to evaluate the last term as follows:

$$\int_{K(y)} r^{2(m-n)}\,dx = \int_\Omega \int_0^\rho r^{2m-n-1}\,dr\,d\omega = \frac{\gamma n \rho^{2m-n}}{\omega_n(2m-n)} \ .$$

Thus we have

$$|u(y)|^2 \le C_{(m,n)}\rho^{2m-n} \int_{K(y)} |D_r^m(\tau u)|^2\,dx \tag{4.2}$$

where $C_{(m,n)}$ is a constant depending only on m and n. From the estimate (4.1) and the formulas for derivatives of a product we obtain

$$\begin{aligned} |D_r^m(\tau u)| &= \Big|\sum_{k=0}^{m} \binom{n}{k} D_r^{m-k}\tau \cdot D_r^k u\Big| \\ &\le \sum_{k=0}^{m} \binom{n}{k} \frac{A_{m-k}}{\rho^{m-k}} |D_r^k u| \ , \end{aligned}$$

hence,

$$|D_r^m(\tau u)|^2 \le C' \sum_{k=0}^{m} \frac{1}{\rho^{2(m-k)}} |D_r^k u|^2 \ .$$

This gives with (4.2) the estimate

$$|u(y)|^2 \le C(m,n)C' \sum_{k=0}^{m} \rho^{2k-n} \int_{K(y)} |D_r^k u|^2\,dx \ . \tag{4.3}$$

By the chain rule we have

$$|D_r^k u|^2 \le C'' \sum_{|\alpha|\le k} |D^\alpha u(x)|^2 \ ,$$

so by extending the integral in (4.3) to all of G we obtain

$$\sup_{y\in G} |u(y)| \le C\|u\|_m \ . \tag{4.4}$$

This proves the following.

Theorem 4.1 *Let G be a bounded open set in $\mathbb{R}^n$ and assume G satisfies the cone condition. Then for every $u \in C^m(\bar{G})$ with $m > n/2$ the estimate (4.4) holds.*

The inequality (4.4) gives us an imbedding theorem. We let $C_u(G)$ denote the linear space of all uniformly continuous functions on G. Then

$$\|u\|_{\infty,0} \equiv \sup\{|u(x)| : x \in G\}$$

is a norm on $C_u(G)$ for which it is a Banach space, i.e., complete. Similarly,

$$\|u\|_{\infty,k} \equiv \sup\{|D^\alpha u(x)| : x \in G \ , \ |\alpha| \leq k\}$$

is a norm on the linear space $C_u^k(G) = \{u \in C_u(G) : D^\alpha \in C_u(G)$ for $|\alpha| \leq k\}$ and the resulting normed linear space is complete.

Theorem 4.2 *Let G be a bounded open set in $\mathbb{R}^n$ and assume G satisfies the cone condition. Then $H^m(G) \subset C_u^k(G)$ where m and k are integers with $m > k + n/2$. That is, each $u \in H^m(G)$ is equal a.e. to a unique function in $C_u^k(G)$ and this identification is continuous.*

Proof: By applying (4.4) to $D^\alpha u$ for $|\alpha| \leq k$ we obtain

$$\|u\|_{\infty,k} \leq C\|u\|_m \ , \qquad u \in C^m(\bar{G}) \ . \tag{4.5}$$

Thus, the identity is continuous from the dense subset $C^m(\bar{G})$ of $H^m(G)$ into the Banach space $C_u^k(G)$. The desired result follows from Theorem I.3.1 and the identification of $H^m(G)$ in $L^2(G)$ (cf. Theorem 2.1).

5 Density and Compactness

The complementary results on Sobolev spaces that we obtain below will be used in later sections. We first show that if $\partial^\alpha f \in L^2(G)$ for all α with $|\alpha| \leq m$, and if ∂G is sufficiently smooth, then $f \in H^m(G)$. The second result is that the injection $H^{m+1}(G) \to H^m(G)$ is a compact mapping.

5.1

We first consider the set $\mathcal{H}^m(G)$ of all $f \in L^2(G)$ for which $\partial^\alpha f \in L^2(G)$ for all α with $|\alpha| \leq m$. It follows easily that $\mathcal{H}^m(G)$ is a Hilbert space with the scalar product and norm as defined on $H^m(G)$ and that $H^m(G) \leq \mathcal{H}^m(G)$. Our plan is to show equality holds when G has a smooth boundary. The case of empty ∂G is easy.

Lemma 5.1 *$C_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{H}^m(\mathbb{R}^n)$.*

The proof of this is similar to that of Theorem 2.3 and we leave it as an exercise. Next we obtain our desired result for the case of ∂G being a hyperplane.

Lemma 5.2 *$H^m(\mathbb{R}^n_+) = \mathcal{H}^m(\mathbb{R}^n_+)$.*

Proof: We need to show each $u \in \mathcal{H}^m(\mathbb{R}^n_+)$ can be approximated from $C^m\overline{(\mathbb{R}^n_+)}$. Let $\varepsilon > 0$ and define $u_\varepsilon(x) = u(x', x_n + \varepsilon)$ for $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$, $x_n > -\varepsilon$. We have $u_\varepsilon \to u$ in $\mathcal{H}^m(\mathbb{R}^n_+)$ as $\varepsilon \to 0$, so it suffices to show $u_\varepsilon \in H^m(\mathbb{R}^n_+)$. Let $\theta \in C^\infty(\mathbb{R})$ be monotone with $\theta(x) = 0$ for $x \leq -\varepsilon$ and $\theta(x) = 1$ for $x > 0$. Then the function θu_ε given by $\theta(x_n)u_\varepsilon(x)$ for $x_n > -\varepsilon$ and by 0 for $x_n \leq -\varepsilon$, belongs to $\mathcal{H}^m(\mathbb{R}^n)$ and clearly $\theta u_\varepsilon = u_\varepsilon$ on $\mathbb{R}^n_+$. Now use Lemma 5.1 to obtain a sequence $\{\varphi_n\}$ from $C_0^\infty(\mathbb{R}^n)$ converging to θu_ε in $\mathcal{H}^m(\mathbb{R}^n)$. The restrictions $\{\varphi_n|_{\mathbb{R}^n_+}\}$ belong to $C^\infty\overline{(\mathbb{R}^n_+)}$ and converge to θu_ε in $\mathcal{H}^m(\mathbb{R}^n_+)$.

Lemma 5.3 *There exists an operator $\mathcal{P} \in \mathcal{L}(\mathcal{H}^m(\mathbb{R}^n_+), \mathcal{H}^m(\mathbb{R}^n))$ such that $(\mathcal{P}u)(x) = u(x)$ for a.e. $x \in \mathbb{R}^n_+$.*

Proof: By Lemma 5.2 it suffices to define such a $\mathcal{P}$ on $C^m\overline{(\mathbb{R}^n_+)}$. Let the numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the solution of the system

$$\begin{cases} \lambda_1 + \lambda_2 + \cdots + \lambda_m = 1 \\ -(\lambda_1 + \lambda_2/2 + \cdots + \lambda_m/m) = 1 \\ \quad\cdots\cdots \\ (-1)^{m-1}(\lambda_1 + \lambda_2/2^{m-1} + \cdots + \lambda_m/m^{m-1}) = 1 \end{cases} . \tag{5.1}$$

For each $u \in C^m\overline{(\mathbb{R}^n_+)}$ we define

$$\mathcal{P}u(x) = \begin{cases} u(x) \;, & x_n \geq 0 \\ \lambda_1 u(x', -x_n) + \lambda_2 u\left(x', -\dfrac{x_n}{2}\right) + \cdots + \lambda_m u\left(x', -\dfrac{x_n}{m}\right) \;, & x_n < 0 \;. \end{cases}$$

The equations (5.1) are precisely the conditions that $\partial_n^j(\mathcal{P}u)$ is continuous at $x_n = 0$ for $j = 0, 1, \ldots, m-1$. From this follows $\mathcal{P}u \in \mathcal{H}^m(\mathbb{R}^n)$; $\mathcal{P}$ is clearly linear and continuous.

Theorem 5.4 *Let G be a bounded open set in $\mathbb{R}^n$ which lies on one side of its boundary, ∂G, which is a C^m-manifold. Then there exists an operator $\mathcal{P}_G \in \mathcal{L}(\mathcal{H}^m(G), \mathcal{H}^m(\mathbb{R}^n))$ such that $(\mathcal{P}_G u)|_G = u$ for every $u \in \mathcal{H}^m(G)$.*

Proof: Let $\{(\varphi_k, G_k) : 1 \le k \le N\}$ be coordinate patches on ∂G and let $\{\beta_k : 0 \le k \le N\}$ be the partition-of-unity constructed in Section 2.3. Thus for each $u \in \mathcal{H}^m(G)$ we have $u = \sum_{j=0}^{N}(\beta_j u)$. The first term $\beta_0 u$ has a trivial extension to an element of $\mathcal{H}^m(\mathbb{R}^n)$. Let $1 \le k \le N$ and consider $\beta_k u$. The coordinate map $\varphi_k : Q \to G_k$ induces an isomorphism $\varphi_k^* : \mathcal{H}^m(G_k \cap G) \to \mathcal{H}^m(Q_+)$ by $\varphi_k^*(v) = v \circ \varphi_k$. The support of $\varphi_k^*(\beta_k u)$ is inside Q so we can extend it as zero in $\mathbb{R}^n_+ \sim Q$ to obtain an element of $\mathcal{H}^m(\mathbb{R}^n_+)$. By Lemma 5.3 this can be extended to an element $\mathcal{P}(\varphi_k^*(\beta_k u))$ of $\mathcal{H}^m(\mathbb{R}^n)$ with support in Q. (Check the proof of Lemma 5.3 for this last claim.) The desired extension of $\beta_k u$ is given by $\mathcal{P}(\varphi_k^*(\beta_k u)) \circ \varphi_k^{-1}$ extended as zero off of G_k. Thus we have the desired operator given by

$$\mathcal{P}_G u = \beta_0 u + \sum_{k=1}^{N} (\mathcal{P}(\beta_k u) \circ \varphi_k) \circ \varphi_i^{-1}$$

where each term is extended as zero as indicated above.

Theorem 5.5 *Let G be given as in Theorem 5.4. Then $H^m(G) = \mathcal{H}^m(G)$.*

Proof: Let $u \in \mathcal{H}^m(G)$. Then $\mathcal{P}_G u \in \mathcal{H}^m(\mathbb{R}^n)$ and Lemma 5.1 gives a sequence $\{\varphi_n\}$ in $C_0^\infty(\mathbb{R}^n)$ which converges to $\mathcal{P}_G u$. Thus, $\{\varphi_n|_G\}$ converges to u in $\mathcal{H}^m(G)$.

5.2

We recall from Section I.7 that a linear function T from one Hilbert space to another is called *compact* if it is continuous and if the image of any bounded set contains a convergent sequence. The following results will be used in Section III.6 and Theorem III.7.7.

Lemma 5.6 *Let Q be a cube in $\mathbb{R}^n$ with edges of length $d > 0$. If $u \in C^1(\bar{Q})$, then*

$$\|u\|^2_{L^2(Q)} \le d^{-n}\left(\int_Q u\right)^2 + (nd^2/2)\sum_{j=1}^{n} \|D_j u\|^2_{L^2(Q)} \ . \tag{5.2}$$

Proof: For $x, y \in Q$ we have

$$u(x) - u(y) = \sum_{j=1}^{n} \int_{x_j}^{y_j} D_j u(y_1, \ldots, y_{j-1}, s, x_{j+1}, \ldots, x_n)\, ds\ .$$

Square this identity and use Theorem I.4.1(a) to obtain

$$u^2(x) + u^2(y) - 2u(x)u(y) \le nd \sum_{j=1}^{n} \int_{a_j}^{b_j} (D_j u)^2 (y_1, \ldots, y_{j-1}, s, x_{j+1}, \ldots, x_n)\, ds$$

where $Q = \{x : a_j \le x_j \le b_j\}$ and $b_k - a_k = d$ for each $k = 1, 2, \ldots, n$. Integrate the preceding inequality with respect to $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, and we have

$$2d^n \|u\|^2_{L^2(Q)} \le 2\left(\int_Q u\right)^2 + nd^{n+2} \sum_{j=1}^{n} \|D_j u\|^2_{L^2(Q)}\ .$$

The desired estimate (5.2) follows.

Theorem 5.7 *Let G be bounded in $\mathbb{R}^n$. If the sequence $\{u_k\}$ in $H_0^1(G)$ is bounded, then there is a subsequence which converges in $L^2(G)$. That is, the injection $H_0^1(G) \to L^2(G)$ is compact.*

Proof: We may assume each $u_k \in C_0^\infty(G)$; set $M = \sup\{\|u_k\|_{H_0^1}\}$. Enclose G in a cube Q; we may assume the edges of Q are of unit length. Extend each u_k as zero on $Q \sim G$, so each $u_k \in C_0^\infty(Q)$ with $\|u_k\|_{H_0^1(Q)} \le M$.

Let $\varepsilon > 0$. Choose integer N so large that $2nM^2/N^2 < \varepsilon$. Decompose Q into equal cubes Q_j, $j = 1, 2, \ldots, N^n$, with edges of length $1/N$. Since $\{u_k\}$ is bounded in $L^2(Q)$, it follows from Theorem I.6.2 that there is a subsequence (denoted hereafter by $\{u_k\}$) which is weakly convergent in $L^2(Q)$. Thus, there is an integer K such that

$$\left|\int_{Q_j} (u_k - u_\ell)\right|^2 < \varepsilon/2N^{2n}\ , \qquad j = 1, 2, \ldots, N^n\ ;\ k, \ell \ge K\ .$$

If we apply (5.2) on each Q_j with $u = u_k - u_\ell$ and sum over all j's, we obtain for $k, \ell \ge K$

$$\|u_k - u_\ell\|^2_{L^2(Q)} \le N^n \left(\sum_{j=1}^{N^n} \varepsilon/2N^{2n}\right) + (n/2N^2)(2M^2) < \varepsilon\ .$$

Thus, $\{u_k\}$ is a Cauchy sequence in $L^2(Q)$.

Corollary *Let G be bounded in $\mathbb{R}^n$ and let $m \geq 1$. Then the injection $H_0^m(G) \to H_0^{m-1}(G)$ is compact.*

Theorem 5.8 *Let G be given as in Theorem 5.4 and let $m \geq 1$. Then the injection $H^m(G) \to H^{m-1}(G)$ is compact.*

Proof: Let $\{u_k\}$ be bounded in $H^m(G)$. Then the sequence of extensions $\{\mathcal{P}_G(u_k)\}$ is bounded in $H^1(\mathbb{R}^n)$. Let $\theta \in C_0^\infty(\mathbb{R}^n)$ with $\theta \equiv 1$ on G and let Ω be an open bounded set in $\mathbb{R}^n$ containing the support of θ. The sequence $\{\theta \cdot \mathcal{P}_G(u_k)\}$ is bounded in $H_0^m(\Omega)$, hence, has a subsequence (denoted by $\{\theta \cdot \mathcal{P}_G(u_{k'})\}$) which is convergent in $\mathcal{H}_0^{m-1}(\Omega)$. The corresponding subsequence of restrictions to G is just $\{u_{k'}\}$ and is convergent in $H^{m-1}(G)$.

Exercises

1.1. Evaluate $(\partial - \lambda)(H(x)e^{\lambda x})$ and $(\partial^2 + \lambda^2)(\lambda^{-1}H(x)\sin(\lambda x))$ for $\lambda \neq 0$.

1.2. Find all distributions of the form $F(t) = H(t)f(t)$ where $f \in C^2(\mathbb{R})$ such that

$$(\partial^2 + 4)F = c_1\delta + c_2\partial\delta \ .$$

1.3. Let K be the square in $\mathbb{R}^2$ with corners at (1,1), (2,0), (3,1), (2,2), and let T_K be the function equal to 1 on K and 0 elsewhere. Evaluate $(\partial_1^2 - \partial_2^2)T_K$.

1.4. Obtain the results of Section 1.6(e) from those of Section 1.6(d).

1.5. Evaluate $\Delta_n(1/|x|^{n-2})$.

1.6. (a) Let G be given as in Section 1.6(e). Show that for each function $f \in C^1(\bar{G})$ the identity

$$\int_G \partial_j f(x)\,dx = \int_{\partial G} f(s)\nu_j(s)\,ds \ , \qquad 1 \leq j \leq n \ ,$$

follows from the fundamental theorem of calculus.

(b) Show that Green's first identity

$$\int_G (\nabla u \cdot \nabla v + (\Delta_n u)v)\, dx = \int_{\partial G} \frac{\partial u}{\partial v}\, v\, ds$$

follows from above for $u \in C^2(\bar{G})$ and $v \in C^1(\bar{G})$. Hint: Take $f_j = (\partial_j u)v$ and add.

(c) Obtain Green's second identity from above.

2.1. In the Hilbert space $H^1(G)$ show the orthogonal complement of $H_0^1(G)$ is the subspace of those $\varphi \in H^1(G)$ for which $\Delta_n \varphi = \varphi$. Find a basis for $H_0^1(G)^\perp$ in each of the three cases $G = (0,1)$, $G = (0,\infty)$, $G = \mathbb{R}$.

2.2. If $G = (0,1)$, show $H^1(G) \subset C(\bar{G})$.

2.3. Show that $H_0^1(G)$ is a Hilbert space with the scalar product

$$(f,g) = \int_G \nabla f(x) \cdot \overline{\nabla g}(x)\, dx \ .$$

If $F \in L^2(G)$, show $T(v) = (F,v)_{L^2(G)}$ defines $T \in H_0^1(G)'$. Use the second part of the proof of Theorem 2.2 to show that there is a unique $u \in H_0^1(G)$ with $\Delta_n u = F$.

2.4. If $G_1 \subset G_2$, show $H_0^m(G_1)$ is naturally identified with a closed subspace of $H_0^m(G_2)$.

2.5. If $u \in H^m(G)$, then $\beta \in C^\infty(\bar{G})$ implies $\beta u \in H^m(G)$, and $\beta \in C_0^\infty(G)$ implies $\beta u \in H_0^m(G)$.

2.6. In the situation of Section 2.3, show that $\|u\|_{H^m(G)}$ is equivalent to $(\sum_{j=0}^N \|\beta_j u\|^2_{H^m(G\cap G_j)})^{1/2}$ and that $\|u\|_{L^2(\partial G)}$ is equivalent to $(\sum_{j=1}^N \|\beta_j u\|^2_{L^2(\partial G\cap G_j)})^{1/2}$.

3.1. In the proof of Theorem 3.2, explain why $\gamma_0(u) = 0$ implies $u(x',s) = \int_0^s \partial_n u(x',t)\, dt$ for a.e. $x' \in \mathbb{R}^{n-1}$.

3.2. Provide all remaining details in the proof of Theorem 3.3.

3.3. Extend the first and second Green's identities to pairs of functions from appropriate Sobolev spaces. (Cf. Section 1.6(e) and Exercise 1.6).

4.1. Show that G satisfies the cone condition if ∂G is a C^1-manifold of dimension $n-1$.

4.2. Show that G satisfies the cone condition if it is convex.

4.3. Show $H^m(G) \subset C^k(G)$ for any open set in $\mathbb{R}^n$ so long as $m > k + n/2$. If $x_0 \in G$, show that $\delta(\varphi) = \overline{\varphi(x_0)}$ defines $\delta \in H^m(G)'$ for $m > n/2$.

4.4. Let Γ be a subset of ∂G in the situation of Theorem 3.3. Show that $\varphi \to \int_\Gamma g(s)\varphi(s)\,ds$ defines an element of $H^1(G)'$ for each $g \in L^2(\Gamma)$. Repeat the above for an $(n-1)$-dimensional C^1-manifold in $\bar{G}$, not necessarily in ∂G.

5.1. Verify that $\mathcal{H}^m(G)$ is a Hilbert space.

5.2. Prove Lemma 5.1.

Chapter III

Boundary Value Problems

1 Introduction

We shall recall two classical boundary value problems and show that an appropriate generalized or abstract formulation of each of these is a well-posed problem. This provides a weak global solution to each problem and motivates much of our latter discussion.

1.1

Suppose we are given a subset G of $\mathbb{R}^n$ and a function $F : G \to \mathbb{K}$. We consider two boundary value problems for the partial differential equation

$$-\Delta_n u(x) + u(x) = F(x) \ , \qquad x \in G \ . \tag{1.1}$$

The *Dirichlet problem* is to find a solution of (1.1) for which $u = 0$ on ∂G. The *Neumann problem* is to find a solution of (1.1) for which $(\partial u/\partial \nu) = 0$ on ∂G. In order to formulate these problems in a meaningful way, we recall the first formula of Green

$$\int_G ((\Delta_n u)v + \nabla u \cdot \nabla v) = \int_{\partial G} \frac{\partial u}{\partial \nu} v = \int_{\partial G} \gamma_1 u \cdot \gamma_0 v \tag{1.2}$$

which holds if ∂G is sufficiently smooth and if $u \in H^2(G)$, $v \in H^1(G)$. Thus, if u is a solution of the Dirichlet problem and if $u \in H^2(G)$, then we have $u \in H_0^1(G)$ (since $\gamma_0 u = 0$) and (from (1.1) and (1.2))

$$(u, v)_{H^1(G)} = (F, v)_{L^2(G)} \ , \qquad v \in H_0^1(G) \ . \tag{1.3}$$

Note that the identity (1.3) holds in general only for those $v \in H^1(G)$ for which $\gamma_0 v = 0$. If we drop the requirement that v vanish on ∂G, then there would be a contribution from (1.2) in the form of a boundary integral. Similarly, if u is a solution of the Neumann problem and $u \in H^2(G)$, then (since $\gamma_1 u = 0$) we obtain from (1.1) and (1.2) the identity (1.3) for all $v \in H^1(G)$. That is, $u \in H^2(G)$ and (1.3) holds for all $v \in H^1(G)$.

Conversely, suppose $u \in H^2(G) \cap H_0^1(G)$ and (1.3) holds for all $v \in H_0^1(G)$. Then (1.3) holds for all $v \in C_0^\infty(G)$, so (1.1) is satisfied in the sense of distributions on G, and $\gamma_0 u = 0$ is a boundary condition. Thus, u is a solution of a Dirichlet problem. Similarly, if $u \in H^2(G)$ and (1.3) holds for all $v \in H^1(G)$, then $C_0^\infty(G) \subset H^1(G)$ shows (1.1) is satisfied as before, and substituting (1.1) into (1.3) gives us

$$\int_{\partial G} \gamma_1 u \cdot \gamma_0 v = 0 \ , \qquad v \in H^1(G) \ .$$

Since the range of γ_0 is dense in $L^2(\partial G)$, this implies that $\gamma_1 u = 0$, so u is a solution of a Neumann problem.

1.2

The preceding remarks suggest a weak formulation of the Dirichlet problem as follows:

> Given $F \in L^2(G)$, find $u \in H_0^1(G)$ such that (1.3) holds for all $u \in H_0^1(G)$.

In particular, the condition that $u \in H^2(G)$ is not necessary for this formulation to make sense. A similar formulation of the Neumann problem would be the following:

> Given $F \in L^2(G)$, find $u \in H^1(G)$ such that (1.3) holds for all $v \in H^1(G)$.

This formulation does not require that $u \in H^2(G)$, so we do not necessarily have $\gamma_1 u \in L^2(\partial G)$. However, we can either extend the operator γ_1 so (1.2) holds on a larger class of functions, or we may prove a regularity result to the effect that a solution of the Neumann problem is necessarily in $H^2(G)$. We shall achieve both of these in the following, but for the present we consider the following abstract problem:

Given a Hilbert space V and $f \in V'$, find $u \in V$ such that for all $v \in V$

$$(u, v)_V = f(v) \ .$$

By taking $V = H_0^1(G)$ or $V = H^1(G)$ and defining f to be the functional $f(v) = (F, v)_{L^2(G)}$ of V', we recover the weak formulations of the Dirichlet or Neumann problems, respectively. But Theorem I.4.5 shows that this problem is well-posed.

Theorem 1.1 *For each $f \in V'$, there exists exactly one $u \in V$ such that $(u, v)_V = f(v)$ for all $v \in V$, and we have $\|u\|_V = \|f\|_{V'}$.*

Corollary *If u_1 and u_2 are the solutions corresponding to f_1 and f_2, then*

$$\|u_1 - u_2\|_V = \|f_1 - f_2\|_{V'} \ .$$

Finally, we note that if $V = H_0^1(G)$ or $H^1(G)$, and if $F \in L^2(G)$ then $\|f\|_{V'} \leq \|F\|_{L^2(G)}$ where we identify $L^2(G) \subset V'$ as indicated.

2 Forms, Operators and Green's Formula

2.1

We begin with a generalization of the weak Dirichlet problem and of the weak Neumann problem of Section 1:

Given a Hilbert space V, a continuous sesquilinear form $a(\cdot, \cdot)$ on V, and $f \in V'$, find $u \in V$ such that

$$a(u, v) = f(v) \ , \qquad v \in V \ . \tag{2.1}$$

The sesquilinear form $a(\cdot, \cdot)$ determines a pair of operators $\alpha, \beta \in \mathcal{L}(V)$ by the identities

$$a(u, v) = (\alpha(u), v)_V = (u, \beta(v))_V \ , \qquad u, v \in V \ . \tag{2.2}$$

Theorem I.4.5 is used to construct α and β from $a(\cdot, \cdot)$, and $a(\cdot, \cdot)$ is clearly determined by either of α or β through (2.2). Theorem I.4.5 also defines the bijection $J \in \mathcal{L}(V', V)$ for which

$$f(v) = (J(f), v)_V \ , \qquad f \in V' \ , \ v \in V \ .$$

In fact, J is just the inverse of R_V. It is clear that u is a solution of the "weak" problem associated with (2.1) if and only if $\alpha(u) = J(f)$. Since J is a bijection, the solvability of this functional equation in V depends on the invertibility of the operator α. A useful sufficient condition for α to be a bijection is given in the following.

Definition. The sesquilinear form $a(\cdot,\cdot)$ on the Hilbert space V is V-*coercive* if there is a $c > 0$ such that

$$|a(v,v)| \geq c\|v\|_V^2 \ , \qquad v \in V \ . \tag{2.3}$$

We show that the weak problem associated with a V-coercive form is well-posed.

Theorem 2.1 *Let $a(\cdot,\cdot)$ be a V-coercive continuous sesquilinear form. Then, for every $f \in V'$, there is a unique $u \in V$ for which* (2.1) *is satisfied. Furthermore, $\|u\|_V \leq (1/c)\|f\|_{V'}$.*

Proof: The estimate (2.3) implies that both α and β are injective, and we also obtain

$$\|\alpha(v)\|_V \geq c\|v\|_V \ , \qquad v \in V \ .$$

This estimate implies that the range of α is closed. But β is the adjoint of α in V, so the range of α, $\mathrm{Rg}(\alpha)$, satisfies the orthogonality condition $\mathrm{Rg}(\alpha)^\perp = K(\beta) = \{0\}$. Hence, $\mathrm{Rg}(\alpha)$ is dense in V, and this shows $\mathrm{Rg}(\alpha) = V$. Since J is norm-preserving the stated results follow easily.

2.2

We proceed now to construct some operators which characterize solutions of problem (2.1) as solutions of boundary value problems for certain choices of $a(\cdot,\cdot)$ and V. First, define $\mathcal{A} \in \mathcal{L}(V,V')$ by

$$a(u,v) = \mathcal{A}u(v) \ , \qquad u,v \in V \ . \tag{2.4}$$

There is a one-to-one correspondence between continuous sesquilinear forms on V and linear operators from V to V', and it is given by the identity (2.4). In particular, u is a solution of the weak problem (2.1) if and only if $u \in V$ and $\mathcal{A}u = f$, so the problem is determined by $\mathcal{A}$ when $f \in V'$ is regarded as data. We would like to know that the identity $\mathcal{A}u = f$ implies that u satisfies a partial differential equation. It will not be possible in all of our examples to identify V' with a space of distributions on a domain G in $\mathbb{R}^n$. (For example, we are thinking of $V = H^1(G)$ in a Neumann problem as in (1.1). The difficulty is that the space $C_0^\infty(G)$ is *not dense* in V.)

There are two "natural" ways around this difficulty. First, we assume there is a Hilbert space H such that V is dense and continuously imbedded in H (hence, we may identify $H' \subset V'$) and such that H is identified with H' through the Riesz map. Thus we have the inclusions

$$V \hookrightarrow H = H' \hookrightarrow V'$$

and the identity

$$f(v) = (f, v)_H \ , \qquad f \in H \ , \ v \in V \ . \tag{2.5}$$

We call H the *pivot space* when we identify $H = H'$ as above. (For example, in the Neumann problem of Section 1, we choose $H = L^2(G)$, and for this choice of H, the Riesz map is the identification of functions with functionals which is compatible with the identification of $L^2(G)$ as a space of distributions on G; cf., Section I.5.3.) We define $D = \{u \in V : \mathcal{A}u \in H\}$. In the examples, $\mathcal{A}u = f$, $u \in D$, will imply that a partial differential equation is satisfied, since $C_0^\infty(G)$ will be dense in H. Note that $u \in D$ if and only if $u \in V$ and there is a $K > 0$ such that

$$|a(u, v)| \leq K\|v\|_H \ , \qquad v \in V \ .$$

(This follows from Theorem I.4.5.) Finally, we obtain the following result.

Theorem 2.2 *If $a(\cdot, \cdot)$ is V-coercive, then D is dense in V, hence, dense in H.*

Proof: Let $w \in V$ with $(u, w)_V = 0$ for all $u \in D$. Then the operator β from (2.2) being surjective implies $w = \beta(v)$ for some $v \in V$. Hence, we obtain $0 = (u, \beta(v))_V = \mathcal{A}u(v) = (\mathcal{A}u, v)_H$ by (2.5), since $u \in D$. But $\mathcal{A}$ maps D onto H, so $v = 0$, hence, $w = 0$.

A second means of obtaining a partial differential equation from the continuous sesquilinear form $a(\cdot, \cdot)$ on V is to consider a closed subspace V_0 of V, let $i : V_0 \hookrightarrow V$ denote the identity and $\rho = i' : V' \to V_0'$ the restriction to V_0 of functionals on V, and define $A = \rho\mathcal{A} : V \to V_0'$. The operator $A \in \mathcal{L}(V, V_0')$ defined by

$$a(u, v) = Au(v) \ , \qquad u \in V \ , \ v \in V_0$$

is called the *formal operator* determined by $a(\cdot,\cdot)$, V and V_0. In examples, V_0 will be the closure in V of $C_0^\infty(G)$, so V_0' is a space of distributions on G. Thus, $Au = f \in V_0'$ will imply that a partial differential equation is satisfied.

2.3

We shall compare the operators $\mathcal{A}$ and A. Assume V_0 is a closed subspace of V, H is a Hilbert space identified with its dual, the injection $V \hookrightarrow H$ is continuous, and V_0 is dense in H. Let D be given as above and define $D_0 = \{u \in V : Au \in H\}$, where we identify $H \subset V_0'$. Note that $u \in D_0$ if and only if $u \in V$ and there is a $K > 0$ such that

$$|a(u, v)| \leq K\|v\|_H \ , \qquad v \in V_0 \ ,$$

so $D \subset D_0$. It is on D_0 that we compare $\mathcal{A}$ and A. So, let $u \in D_0$ be fixed in the following and consider the functional

$$\varphi(v) = \mathcal{A}u(v) - (Au, v)_H \ , \qquad v \in V \ . \tag{2.6}$$

Then we have $\varphi \in V'$ and $\varphi|_{V_0} = 0$. But these are precisely the conditions that characterize those $\varphi \in V'$ which are in the range of $q' : (V/V_0)' \to V'$, the dual of the quotient map $q : V \to V/V_0$. That is, there is a unique $F \in (V/V_0)'$ such that $q'(F) = F \circ q = \varphi$. Thus, (2.6) determines an $F \in (V/V_0)'$ such that $F(q(v)) = \varphi(v)$, $v \in V$. In order to characterize (V/V_0), let V_0 be the kernel of a linear surjection $\gamma : V \to B$ and denote by $\hat{\gamma}$ the quotient map which is a bijection of V/V_0 onto B. Define a norm on B by $\|\hat{\gamma}(\hat{x})\|_B = \|\hat{x}\|_{V/V_0}$ so $\hat{\gamma}$ is bicontinuous. Then the dual operator $\hat{\gamma}' : B' \to (V/V_0)'$ is a bijection. Given the functional F above, there is a unique $\partial \in B'$ such that $F = \hat{\gamma}'(\partial)$. That is, $F = \partial \circ \hat{\gamma}$. We summarize the preceding discussion in the following result.

Theorem 2.3 *Let V and H be Hilbert spaces with V dense and continuously imbedded in H. Let H be identified with its dual H' so* (2.5) *holds. Suppose γ is a linear surjection of V onto a Hilbert space B such that the quotient map $\hat{\gamma} : V/V_0 \to B$ is norm-preserving, where V_0, the kernel of γ, is dense in H.*

Thus, we have $V_0 \hookrightarrow H \hookrightarrow V_0'$. *Let* $\mathcal{A} \in \mathcal{L}(V, V')$ *and define* $A \in \mathcal{L}(V, V_0')$ *by* $A = \rho\mathcal{A}$, *where* $\rho : V' \to V_0'$ *is restriction to* V_0, *the dual of the injection* $V_0 \hookrightarrow V$. *Let* $D_0 = \{u \in V : Au \in H\}$. *Then, for every* $u \in D_0$, *there is a unique* $\partial(u) \in B'$ *such that*

$$\mathcal{A}u(v) - (Au, v)_H = \partial(u)(\gamma(v)) \ , \qquad v \in V \ . \tag{2.7}$$

The mapping $\partial : D_0 \to B'$ *is linear.*

When V_0' is a space of distributions, it is the formal operator A that determines a partial differential equation. When γ is a trace function and V_0 consists of those elements of V which vanish on a boundary, the quotient V/V_0 represents boundary values of elements of V. Thus B is a realization of these abstract boundary values as a function space and (2.7) is an abstract Green's formula. We shall call ∂ the abstract *Green's operator*.

Example. Let $V = H^1(G)$ and $\gamma : H^1(G) \to L^2(\partial G)$ be the trace map constructed in Theorem II.3.1. Then $H_0^1(G) = V_0$ is the kernel of γ and we denote by B the range of γ. Since $\hat{\gamma}$ is norm-preserving, the injection $B \hookrightarrow L^2(\partial G)$ is continuous and, by duality, $L^2(\partial G) \subset B'$, where we identify $L^2(\partial G)$ with its dual space. In particular, B consists of functions on ∂G and $L^2(\partial G)$ is a subspace of B'. Continuing this example, we choose $H = L^2(G)$ and $a(u,v) = (u,v)_{H^1(G)}$, so $Au = -\Delta_n u + u$ and $D_0 = \{u \in H^1(G) : \Delta_n u \in L^2(G)\}$. By comparing (2.7) with (1.2) we find that when ∂G is smooth $\partial : D_0 \to B'$ is an extension of $\partial/\partial\nu = \gamma_1 : H^2(G) \to L^2(\partial G)$.

3 Abstract Boundary Value Problems

3.1

We begin by considering an abstract "weak" problem (2.1) motivated by certain carefully chosen formulations of the Dirichlet and Neumann problems for the Laplace differential operator. The sesquilinear form $a(\cdot,\cdot)$ led to two operators: $\mathcal{A}$, which is equivalent to $a(\cdot,\cdot)$, and the formal operator A, which is determined by the action of $\mathcal{A}$ restricted to a subspace V_0 of V. It is A that will be a partial differential operator in our applications, and its domain will be determined by the space V and the difference of $\mathcal{A}$ and A as characterized by the Green's operator ∂ in Theorem 2.3. If V is prescribed by boundary conditions, then these same boundary conditions will be forced on a solution

u of (2.1). Such boundary conditions are called *stable* or *forced* boundary conditions. A second set of constraints may arise from Theorem 2.3 and these are called *unstable* or *variational* boundary conditions. The complete set of both stable and unstable boundary conditions will be part of the characterization of the domain of the operator A.

We shall elaborate on these remarks by using Theorem 2.3 to characterize solutions of (2.1) in a setting with more structure than assumed before. This additional structure consists essentially of splitting the form $a(\cdot,\cdot)$ into the sum of a spatial part which determines the partial differential equation in the region and a second part which contributes only boundary terms. The functional f is split similarly into a spatial part and a boundary part.

3.2

We assume that we have a Hilbert space V and a linear surjection $\gamma : V \to B$ with kernel V_0 and that B is a Hilbert space isomorphic to V/V_0. Let V be continuously imbedded in a Hilbert space H which is the pivot space identified with its dual, and let V_0 be dense in H. Thus we have the continuous injections $V_0 \hookrightarrow H \hookrightarrow V_0'$ and $V \hookrightarrow H \hookrightarrow V'$ and the identity (2.5). Let $a_1 : V \times V \to \mathbb{K}$ and $a_2 : B \times B \to \mathbb{K}$ be continuous sesquilinear forms and define

$$a(u,v) = a_1(u,v) + a_2(\gamma u, \gamma v) \ , \qquad u,v \in V \ .$$

Similarly, let $F \in H$, $g \in B'$, and define

$$f(v) = (F,v)_H + g(\gamma v) \ , \qquad v \in V \ .$$

The problem (2.1) is the following: find $u \in V$ such that

$$a_1(u,v) + a_2(\gamma u, \gamma v) = (F,v)_H + g(\gamma v) \ , \qquad v \in V \ . \tag{3.1}$$

We shall use Theorem 2.3 to show that (3.1) is equivalent to an abstract boundary value problem.

Theorem 3.1 *Assume we are given the Hilbert spaces, sesquilinear forms and functionals as above. Let $\mathcal{A}_2 : B \to B'$ be given by*

$$\mathcal{A}_2\varphi(\psi) = a_1(\varphi,\psi) \ , \qquad \varphi,\psi \in B \ ,$$

and $A : V \to V_0'$ by

$$Au(v) = a_1(u,v) \ , \qquad u \in V \ , \ v \in V_0 \ .$$

Let $D_0 = \{u \in V : Au \in H\}$ *and* $\partial_1 \in L(D_0, B')$ *be given by (Theorem 2.3)*

$$a_1(u,v) - (Au,v)_H = \partial_1 u(\gamma v) \ , \qquad u \in D_0 \ , \ v \in V \ . \tag{3.2}$$

Then, u *is a solution of* (3.1) *if and only if*

$$u \in V \ , \quad Au = F \ , \quad \partial_1 u + \mathcal{A}_2(\gamma u) = g \ . \tag{3.3}$$

Proof: Since $a_2(\gamma u, \gamma v) = 0$ for all $v \in V_0$, it follows that the formal operator A and space D_0 (determined above by $a_1(\cdot,\cdot)$) are equal, respectively, to the operator and domain determined by $a(\cdot,\cdot)$ in Section 2.3. Suppose u is a solution of (3.1). Then $u \in V$, and the identity (3.1) for $v \in V_0$ and V_0 being dense in H imply that $Au = F \in H$. This shows $u \in D_0$ and using (3.2) in (3.1) gives

$$\partial_1 u(\gamma v) + a_2(\gamma u, \gamma v) = g(\gamma v) \ , \qquad v \in V \ .$$

Since γ is a surjection, this implies the remaining equation in (3.3). Similarly, (3.3) implies (3.1).

Corollary 3.2 *Let* D *be the space of those* $u \in V$ *such that for some* $F \in H$

$$a(u,v) = (F,v)_H \ , \qquad v \in V \ .$$

Then $u \in D$ *if and only if* u *is a solution of* (3.3) *with* $g = 0$.

Proof: Since V_0 is dense in H, the functional $f \in V'$ defined above is in H if and only if $g = 0$.

4 Examples

We shall illustrate some applications of our preceding results in a variety of examples of boundary value problems. Our intention is to indicate the types of problems which can be described by Theorem 3.1.

4.1

Let there be given a set of (coefficient) functions

$$a_{ij} \in L^\infty(G) \ , \quad 1 \le i \ , \ j \le n \ ; \qquad a_j \in L^\infty(G) \ , \quad 0 \le j \le n \ ,$$

where G is open and connected in $\mathbb{R}^n$, and define

$$a(u,v) = \int_G \Big\{ \sum_{i,j=1}^n a_{ij}(x)\partial_i u(x) \partial_j \overline{v(x)} + \sum_{j=0}^n a_j(x)\partial_j u(x)\overline{v(x)} \Big\} \, dx \ , \qquad u,v \in H^1(G) \ , \tag{4.1}$$

where $\partial_0 u = u$. Let $F \in L^2(G) \equiv H$ be given and define $f(v) = (F,v)_H$. Let Γ be a closed subset of ∂G and define

$$V = \{ v \in H^1(G) : \gamma_0(v)(s) = 0 \ , \quad \text{a.e.} \quad s \in \Gamma \} \ .$$

V is a closed subspace of $H^1(G)$, hence a Hilbert space. We let $V_0 = H_0^1(G)$ so the formal operator $A : V \to V_0' \subset \mathcal{D}^*(G)$ is given by

$$Au = - \sum_{i,j=1}^n \partial_j(a_{ij}\partial_i u) + \sum_{j=0}^n a_j \partial_j u \ .$$

Let γ be the restriction to V of the trace map $H^1(G) \to L^2(\partial G)$, where we assume ∂G is appropriately smooth, and let B be the range of γ, hence $B \hookrightarrow L^2(\partial G \sim \Gamma) \hookrightarrow B'$. If all the $a_{ij} \in C^1(\bar{G})$, then we have from the classical Green's theorem

$$a(u,v) - (Au,v)_H = \int_{\partial G \sim \Gamma} \frac{\partial u}{\partial \nu_A} \cdot \gamma_0(v) \, ds \ , \qquad u \in H^2(G) \ , \ v \in V$$

where

$$\frac{\partial u}{\partial \nu_A} = \sum_{i=1}^n \partial_i u(s) \sum_{j=1}^n a_{ij}(s)\nu_j(s)$$

denotes the (weighted) normal derivative on $\partial G \sim \Gamma$. Thus, the operator ∂ is an extension of $\partial/\partial\nu_A$ from $H^2(G)$ to the domain $D_0 = \{u \in V : Au \in L^2(G)\}$. Theorem 3.1 now asserts that u is a solution of the problem (2.1) if and only if $u \in H^1(G)$, $\gamma_0 u = 0$ on Γ, $\partial u = 0$ on $\partial G \sim \Gamma$, and $Au = F$.

That is, u is a generalized solution of the *mixed Dirichlet-Neumann boundary value problem*

$$\left.\begin{aligned} &Au(x) = F(x)\ , \ x \in G\ ,\\ &u(s) = 0\ , \qquad\quad s \in \Gamma\ ,\\ &\frac{\partial u(s)}{\partial \nu_A} = 0\ , \qquad s \in \partial G \sim \Gamma\ . \end{aligned}\right\} \tag{4.2}$$

If $\Gamma = \partial G$, this is called the Dirichlet problem or the boundary value problem of *first type*. If $\Gamma = \emptyset$, it is called the Neumann problem or boundary value problem of *second type*.

4.2

We shall simplify the partial differential equation but introduce boundary integrals. Define $H = L^2(G)$, $V_0 = H_0^1(G)$, and

$$a_1(u, v) = \int_G \nabla u \cdot \nabla \bar{v} \qquad u, v \in V \tag{4.3}$$

where V is a subspace of $H^1(G)$ to be chosen below. The corresponding distribution-valued operator is given by $A = -\Delta_n$ and ∂_1 is an extension of the standard normal derivative given by

$$\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu\ .$$

Suppose we are given $F \in L^2(G)$, $g \in L^2(\partial G)$, and $\alpha \in L^\infty(\partial G)$. We define

$$\begin{aligned} a_2(\varphi, \psi) &= \int_{\partial G} \alpha(s)\varphi(s)\bar{\psi}(s)\, ds\ , \qquad \varphi, \psi \in L^2(\partial G)\\ f(v) &= (F, v)_H + (g, \gamma_0 v)_{L^2(\partial G)}\ , \qquad v \in V\ , \end{aligned}$$

and then use Theorem 3.1 to characterize a solution u of (2.1) for different choices of V.

If $V = H^1(G)$, then u is a generalized solution of the boundary value problem

$$\left.\begin{aligned} &-\Delta_n u(x) = F(x)\ , \qquad\qquad x \in G\ ,\\ &\frac{\partial u(s)}{\partial \nu} + \alpha(s)u(s) = g(x)\ ,\ s \in \partial G\ . \end{aligned}\right\} \tag{4.4}$$

The boundary condition is said to be of *third type* at those points $s \in \partial G$ where $\alpha(s) \neq 0$.

For an example of *non-local* boundary conditions, choose $V = \{v \in H^1(G) : \gamma_0(v) \text{ is constant}\}$. Let $g(s) = g_0$ and $\alpha(s) = \alpha_0$ be constants, and define $a_2(\cdot,\cdot)$ and f as above. Then u is a solution of the boundary value problem of *fourth type*

$$\left.\begin{aligned} &-\Delta_n u(x) = F(x)\ , && x \in G\ ,\\ &u(s) = u_0 \quad (\text{constant})\ , && s \in \partial G\ ,\\ &\left(\int_{\partial G} \frac{\partial u(s)}{\partial \nu}\, ds \Big/ \int_{\partial G} ds\right) + \alpha_0 \cdot u_0 = g_0\ . && \end{aligned}\right\} \tag{4.5}$$

Note that $B = \mathbb{K}$ in this example and u_0 is not prescribed as data. Also, *periodic* boundary conditions are obtained when G is an interval.

4.3

We consider a problem with a prescribed derivative on the boundary in a direction which is not necessarily normal. For simplicity we assume $n = 2$, let $c \in \mathbb{R}$, and define

$$a(u,v) = \int_G \{\partial_1 u(\partial_1 \bar{v} + c\partial_2 \bar{v}\,) + \partial_2 u(\partial_2 \bar{v} - c\partial_1 \bar{v}\,)\} \tag{4.6}$$

for u, $v \in V = H^1(G)$. Taking $V_0 = H_0^1(G)$ gives $A = -\Delta_2$ and the classical Green's theorem shows that for $u \in H^2(G)$ and $v \in H^1(G)$ we have

$$a(u,v) - (Au,v)_{L^2(G)} = \int_{\partial G} \left(\frac{\partial u}{\partial \nu} + c\frac{\partial u}{\partial \tau}\right) \bar{v}\, ds$$

where

$$\frac{\partial u}{\partial \tau} = \nabla u \cdot \tau$$

is the derivative in the direction of the tangent vector $\tau = (\nu_2, -\nu_1)$ on ∂G. Thus ∂ is an extension of the *oblique derivative* in the direction $\nu + c\tau$ on the boundary. If f is chosen as in (4.2), then Theorem 3.1 shows that problem (2.1) is equivalent to a weak form of the boundary value problem

$$\begin{aligned} &-\Delta_2 u(x) = F(x)\ , && x \in G\ ,\\ &\frac{\partial u}{\partial \nu} + c\frac{\partial u}{\partial \tau} = g(s)\ , && s \in \partial G\ . \end{aligned}$$

4.4

Let G_1 and G_2 be disjoint open connected sets with smooth boundaries ∂G_1 and ∂G_2 which intersect in a C^1 manifold Σ of dimension $n-1$. If ν_1 and ν_2 denote the unit outward normals on ∂G_1 and ∂G_2, then $\nu_1(s) = -\nu_2(s)$ for $s \in \Sigma$. Let G be the interior of the closure of $G_1 \cup G_2$, so that

$$\partial G = \partial G_1 \cup \partial G_2 \sim (\Sigma \sim \partial\Sigma) \ .$$

For $k = 1, 2$, let γ_0^k be the trace map $H^1(G_k) \to L^2(\partial G_k)$. Define $V = H^1(G)$ and note that $\gamma_0^1 u_1(s) = \gamma_0^2 u_2(s)$ for a.e. $s \in \Sigma$ when $u \in H^1(G)$ and u_k is the restriction of u to G_k, $k = 1, 2$. Thus we have a natural trace map

$$\begin{aligned} \gamma : H^1(G) &\longrightarrow L^2(\partial G) \times L^2(\Sigma) \\ u &\longmapsto (\gamma_0 u, \gamma_0^1 u_1|_\Sigma) \ , \end{aligned}$$

where $\gamma_0 u(s) = \gamma_0^k u_k(s)$ for $s \in \partial G_k \sim \Sigma$, $k = 1, 2$, and its kernel is given by $V_0 = H_0^1(G_1) \times H_0^1(G_2)$.

Let $a_1 \in C^1(\bar{G}_1)$, $a_2 \in C^1(\bar{G}_2)$ and define

$$a(u, v) = \int_{G_1} a_1 \nabla u \cdot \nabla \bar{v} + \int_{G_2} a_2 \nabla u \cdot \nabla \bar{v} \ , \qquad u, v \in V \ .$$

The operator A takes values in $\mathcal{D}^*(G_1 \cup G_2)$ and is given by

$$Au(x) = \begin{cases} -\displaystyle\sum_{j=1}^{n} \partial_j (a_1(x) \partial_j u(x)) \ , & x \in G_1 \ , \\ -\displaystyle\sum_{j=1}^{n} \partial_j (a_2(x) \partial_j u(x)) \ , & x \in G_2 \ . \end{cases}$$

The classical Green's formula applied to G_1 and G_2 gives

$$a(u, v) - (Au, v)_{L^2(G)} = \int_{\partial G_1} a_1 \frac{\partial u_1}{\partial \nu_1} \bar{v}_1 + \int_{\partial G_2} a_2 \frac{\partial u_2}{\partial \nu_2} \bar{v}_2$$

for $u \in H^2(G)$ and $v \in H^1(G)$. It follows that the restriction of the operator ∂ to the space $H^2(G)$ is given by $\partial u = (\partial_0 u, \partial_1 u) \in L^2(\partial G) \times L^2(\Sigma)$, where

$$\partial_0 u(s) = a_k(s) \frac{\partial u_k(s)}{\partial \nu_k} \ , \text{ a.e. } \ s \in \partial G_k \sim \Sigma \ , \quad k = 1, 2 \ ,$$

$$\partial_1 u(s) = a_1(s) \frac{\partial u_1(s)}{\partial \nu_1} + a_2(s) \frac{\partial u_2(s)}{\partial \nu_2} \ , \qquad s \in \Sigma \ .$$

Let f be given as in Section 4.2. Then a solution of u of (2.1) is characterized by Theorem 3.1 as a weak solution of the boundary value problem

$$\begin{cases} u_1 \in H^1(G_1)\ ,\ -\sum_{j=1}^{n} \partial_j a_1(x)\partial_j u_1(x) = F(x)\ , & x \in G_1\ , \\ u_2 \in H^1(G_2)\ ,\ -\sum_{j=1}^{n} \partial_j a_2(x)\partial_j u_2(x) = F(x)\ , & x \in G_2\ , \\ a_1(s)\dfrac{\partial u_1(s)}{\partial \nu_1} = g(s)\ , & s \in \partial G_1 \sim \Sigma\ , \\ a_2(s)\dfrac{\partial u_2(s)}{\partial \nu_2} = g(s)\ , & s \in \partial G_2 \sim \Sigma\ , \\ u_1(s) = u_2(s)\ , & \\ a_1(s)\dfrac{\partial u_1(s)}{\partial \nu_1} + a_2(s)\dfrac{\partial u_2(s)}{\partial \nu_2} = 0\ , & s \in \Sigma\ . \end{cases}$$

Since $\nu_1 = -\nu_2$ on Σ, this last condition implies that the normal derivative has a prescribed jump on Σ which is determined by the ratio of $a_1(s)$ to $a_2(s)$. The pair of equations on the interface Σ are known as *transition conditions*.

4.5

Let the sets G_1, G_2 and G be given as in Section 4.4. Suppose Σ_0 is an open subset of the interface Σ which is also contained in the hyperplane $\{x = (x', x_n) : x_n = 0\}$ and define $V = \{v \in H_0^1(G) : \gamma_0^1 u_1|_{\Sigma_0} \in H^1(\Sigma_0)\}$. With the scalar product

$$(u,v)_V \equiv (u,v)_{H_0^1(G)} + (\gamma_0^1 u, \gamma_0^1 v)_{H^1(\Sigma_0)}\ , \qquad u, v \in V\ ,$$

V is a Hilbert space. Let $\gamma(u) = \gamma_0^1(u)|_{\Sigma}$ be the corresponding trace operator $V \to L^2(\Sigma)$, so $K(\gamma) = H_0^1(G_1) \times H_0^1(G_2)$ contains $C_0^\infty(G_1 \cup G_2)$ as a dense subspace. Let $\alpha \in L^\infty(\Sigma_0)$ and define the sesquilinear form

$$a(u,v) = \int_G \nabla u \cdot \nabla \bar{v} + \int_{\Sigma_0} \alpha \nabla'(\gamma u) \cdot \nabla' \overline{(\gamma v)}\ , \qquad u, v \in V\ . \tag{4.7}$$

Where ∇' denotes the gradient in the first $n-1$ coordinates. Then $A = -\Delta_n$ in $\mathcal{D}^*(G_1 \cup G_2)$ and the classical Green's formula shows that ∂u is given by

$$\partial u(v) = \int_\Sigma \left(\frac{\partial u_1}{\partial \nu_1} \bar{v} + \frac{\partial u_2}{\partial \nu_2} \bar{v} \right) + \int_{\Sigma_0} \alpha \nabla'(\gamma(u)) \nabla' \bar{v}$$

for $u \in H^2(G)$ and $v \in B$. Since the range of γ is dense in $L^2(\Sigma \sim \Sigma_0)$, it follows that if $\partial u = 0$ then

$$\frac{\partial u_1(s)}{\partial \nu_1} + \frac{\partial u_2(s)}{\partial \nu_2} = 0 , \qquad s \in \Sigma \sim \Sigma_0 .$$

But $\nu_1 = -\nu_2$ on Σ, so the normal derivative of u is continuous across $\Sigma \sim \Sigma_0$. Since the range of γ contains $C_0^\infty(\Sigma_0)$, it follows that if $\partial u = 0$ then we obtain the identity

$$\int_{\Sigma_0} \alpha \nabla'(\gamma u) \nabla' \overline{(\gamma v)} + \int_{\Sigma_0} \frac{\partial u_1}{\partial \nu_1} \overline{(\gamma v)} + \frac{\partial u_2}{\partial \nu_2} \overline{(\gamma v)} = 0 , \qquad v \in V ,$$

and this shows that $\gamma u|_{\Sigma_0}$ satisfies the abstract boundary value

$$\begin{aligned} -\Delta_{n-1}(\gamma u)(s) &= \frac{\partial u_2(s)}{\partial \nu_1} - \frac{\partial u_1(s)}{\partial \nu_1} , && s \in \Sigma_0 , \\ (\gamma u)(s) &= 0 , && s \in \partial\Sigma_0 \cap \partial G , \\ \frac{\partial(\gamma u)(s)}{\partial \nu_0} &= 0 , && s \in \partial\Sigma_0 \sim \partial G , \end{aligned}$$

where ν_0 is the unit normal on $\partial\Sigma_0$, the $(n-2)$-dimensional boundary of Σ_0.

Let $F \in L^2(G)$ and $f(v) = (F, v)_{L^2(G)}$ for $v \in V$. Then from Corollary 3.2 it follows that (3.3) is a generalized boundary value problem given by

$$\left.\begin{aligned} -\Delta_n u(x) &= F(x) , && x \in G_1 \cup G_2 , \\ u(s) &= 0 , && s \in \partial G , \\ u_1(s) = u_2(s) , \quad \frac{\partial u_1(s)}{\partial \nu_1} &= \frac{\partial u_2(s)}{\partial \nu_1} , && s \in \Sigma \sim \Sigma_0 , \\ -\Delta_{n-1} u(s) &= \frac{\partial u_2(s)}{\partial \nu_1} - \frac{\partial u_1(s)}{\partial \nu_1} , && s \in \Sigma_0 , \\ \frac{\partial u(s)}{\partial \nu_0} &= 0 , && s \in \partial\Sigma_0 \sim \partial G_0 . \end{aligned}\right\} \tag{4.8}$$

Nonhomogeneous terms could be added as in previous examples and similar problems could be solved on interfaces which are not necessarily flat.

5 Coercivity; Elliptic Forms

5.1

Let G be an open set in $\mathbb{R}^n$ and suppose we are given a collection of functions a_{ij}, $1 \le i,\, j \le n$; a_j, $0 \le j \le n$, in $L^\infty(G)$. Define the sesquilinear form

$$a(u,v) = \int_G \left\{ \sum_{i,j=1}^n a_{ij}(x)\partial_i u(x)\partial_j \bar{v}(x) + \sum_{j=0}^n a_j(x)\partial_j u(x)\cdot \overline{v(x)} \right\} dx \qquad (5.1)$$

on $H^1(G)$. We saw in Section 4.1 that such forms lead to partial differential equations of second order on G.

Definition. The sesquilinear form (5.1) is called *strongly elliptic* if there is a constant $c_0 > 0$ such that

$$\mathrm{Re} \sum_{i,j=1}^n a_{ij}(x)\xi_i\bar{\xi_j} \ge c_0 \sum_{j=1}^n |\xi_j|^2\ , \qquad \xi = (\xi_1,\ldots,\xi_n) \in \mathbb{K}^n\ ,\ x \in G\ . \quad (5.2)$$

We shall show that a strongly elliptic form can be made coercive over (any subspace of) $H^1(G)$ by adding a sufficiently large multiple of the identity to it.

Theorem 5.1 *Let* (5.1) *be strongly elliptic. Then there is a* $\lambda_0 \in \mathbb{R}$ *such that for every* $\lambda > \lambda_0$*, the form*

$$a(u,v) + \lambda \int_G u(x)\bar{v}(x)\,dx$$

is $H^1(G)$*-coercive.*

Proof: Let $K_1 = \max\{\|a_j\|_{L^\infty(G)} : 1 \le j \le n\}$ and $K_0 = \operatorname{ess\,inf}\{\mathrm{Re}\, a_0(x) : x \in G\}$. Then, for $1 \le j \le n$ and each $\varepsilon > 0$ we have

$$\begin{aligned} |(a_j\partial_j u, u)_{L^2(G)}| &\le K_1\|\partial_j u\|_{L^2(G)}\cdot\|u\|_{L^2(G)} \\ &\le (K_1/2)\left(\varepsilon\|\partial_j u\|^2_{L^2(G)} + (1/\varepsilon)\|u\|^2_{L^2(G)}\right)\ . \end{aligned}$$

We also have $\mathrm{Re}(a_0 u, u)_{L^2(G)} \ge K_0\|u\|^2_{L^2(G)}$, so using these with (5.2) in (5.1) gives

$$\begin{aligned} \mathrm{Re}\, a(u,u) &\ge (c_0 - \varepsilon K_1/2)\|\nabla u\|^2_{L^2(G)} \\ &\quad + (K_0 - nK_1/2\varepsilon)\|u\|^2_{L^2(G)}\ , \quad u \in H^1(G)\ . \end{aligned} \qquad (5.3)$$

We choose $\varepsilon > 0$ so that $K_1\varepsilon = c_0$. This gives us the desired result with $\lambda_0 = (nK_1^2/2c_0) - K_0$.

Corollary 5.2 *For every $\lambda > \lambda_0$, the boundary value problem* (4.2) *is well-posed, where*

$$Au = -\sum_{i,j=1}^{n} \partial_j(a_{ij}\partial_i u) + \sum_{j=1}^{n} a_j\partial_j u + (a_0 + \lambda)u \ .$$

Thus, for every $F \in L^2(G)$, there is a unique $u \in D$ such that (4.2) *holds, and we have the estimate*

$$\|(\lambda - \lambda_0)u\|_{L^2(G)} \le \|F\|_{L^2(G)} \ . \tag{5.4}$$

Proof: The space D was defined in Section 2.2 and Corollary 3.2, so we need only to verify (5.4). For $u \in D$ and $\lambda > \lambda_0$ we have from (5.3)

$$\begin{aligned}(\lambda - \lambda_0)\|u\|^2_{L^2(G)} &\le a(u,u) + \lambda(u,u)_{L^2(G)} = (Au,u)_{L^2(G)} \\ &\le \|Au\|_{L^2(G)} \cdot \|u\|_{L^2(G)}\end{aligned}$$

and the estimate (5.4) now follows.

5.2

We indicate how coercivity may be obtained from the addition of boundary integrals to strongly elliptic forms.

Theorem 5.3 *Let G be open in $\mathbb{R}^n$ and suppose $0 \le x_n \le K$ for all $x = (x', x_n) \in G$. Let ∂G be a C^1-manifold with G on one side of ∂G. Let $\nu(s) = (\nu_1(s), \ldots, \nu_n(s))$ be the unit outward normal on ∂G and define*

$$\Sigma = \{s \in \partial G : \nu_n(s) > 0\} \ .$$

Then for all $u \in H^1(G)$ we have

$$\int_G |u|^2 \le 2K \int_\Sigma |\gamma_0 u(s)|^2 \, ds + 4K^2 \int_G |\partial_n u|^2 \ .$$

Proof: For $u \in C^1(\bar{G})$, the Gauss Theorem gives

$$\int_{\partial G} \nu_n(s) s_n |u(s)|^2 \, ds = \int_G D_n(x_n |u(x)|^2) \, dx$$
$$= \int_G |u|^2 + \int_G x_n D_n(|u(x)|^2) \, dx \ .$$

Thus, we obtain from the inequality

$$2|a|\,|b| \le \frac{|a|^2}{2K} + 2K|b|^2 \ , \qquad a, b \in \mathbb{C} \ ,$$

the estimate

$$\int_G |u|^2 \le \int_{\partial G} \nu_n s_n |u(s)|^2 \, ds + (1/2) \int_G |u|^2 + 2K^2 \int_G |D_n u|^2 \ .$$

Since $\nu_n(s) s_n \le 0$ for $s \in \partial G \sim \Sigma$, the desired result follows.

Corollary 5.4 *If* (5.1) *is strongly elliptic,* $a_j \equiv 0$ *for* $1 \le j \le n$, $\mathrm{Re}\, a_0(x) \ge 0$, $x \in G$, *and if* $\Sigma \subset \Gamma$, *then the mixed Dirichlet-Neumann problem* (4.2) *is well-posed.*

Corollary 5.5 *If* $\alpha \in L^\infty(\partial G)$ *satisfies*

$$\mathrm{Re}\, \alpha(x) \ge 0 \ , \quad x \in \partial G \ , \quad \mathrm{Re}\, \alpha(x) \ge c > 0 \ , \quad x \in \Sigma \ ,$$

then the third boundary value problem (4.4) *is well-posed. The fourth boundary value problem* (4.5) *is well-posed if* $\mathrm{Re}(\alpha_0) > 0$.

Similar results can be obtained for the example of Section 4.3. Note that the form (4.6) satisfies

$$\mathrm{Re}\, a(u, u) = \int_G \left\{ |\partial_1 u|^2 + |\partial_2 u|^2 \right\} \ , \qquad u \in H^1(G) \ ,$$

so coercivity can be obtained over appropriate subspaces of $H^1(G)$ (as in Corollary 5.4) or by adding a positive multiple of the identity on G or boundary integrals (as in Corollary 5.5). Modification of (4.6) by restricting V, e.g., to consist of functions which vanish on a sufficiently large part of ∂G, or by adding forms, e.g., that are coercive over $L^2(G)$ or $L^2(\partial G)$, will result in a well-posed problem.

Finally, we note that the first term in the form (4.7) is coercive over $H_0^1(G)$ and, hence, over $L^2(\Sigma)$. Thus, if $\mathrm{Re}\, \alpha(x) \ge c > 0$, $x \in \Sigma_0$, then (4.7) is V-coercive and the problem (4.8) is well-posed.

5.3

In order to verify that the sesquilinear forms above were coercive over certain subspaces of $H^1(G)$, we found it convenient to verify that they satisfied the following stronger condition.

Definition. The sesquilinear form $a(\cdot,\cdot)$ on the Hilbert space V is *V-elliptic* if there is a $c > 0$ such that

$$\operatorname{Re} a(v,v) \geq c\|v\|_V^2 \ , \qquad v \in V \ . \tag{5.5}$$

Such forms will occur frequently in our following discussions.

6 Regularity

We begin this section with a consideration of the Dirichlet and Neumann problems for a simple elliptic equation. The original problems were to find solutions in $H^2(G)$ but we found that it was appropriate to seek weak solutions in $H^1(G)$. Our objective here is to show that those weak solutions are in $H^2(G)$ when the domain G and data in the equation are sufficiently smooth. In particular, this shows that the solution of the Neumann problem satisfies the boundary condition in $L^2(\partial G)$ and not just in the sense of the abstract Green's operator constructed in Theorem 2.3, i.e., in B'. (See the Example in Section 2.3.)

6.1

We begin with the Neumann problem; other cases will follow similarly.

Theorem 6.1 *Let G be bounded and open in $\mathbb{R}^n$ and suppose its boundary is a C^2-manifold of dimension $n-1$. Let $a_{ij} \in C^1(G)$, $1 \leq i,\ j \leq n$, and $a_j \in C^1(G)$, $0 \leq j \leq n$, all have bounded derivatives and assume that the sesquilinear form defined by*

$$a(\varphi,\psi) \equiv \int_G \left\{ \sum_{i,j=1}^n a_{ij}\partial_i\varphi\overline{\partial_j\psi} + \sum_{j=0}^n a_j\partial_j\varphi\bar{\psi} \right\} dx \ , \qquad \varphi,\psi \in H^1(G) \tag{6.1}$$

is strongly elliptic. Let $F \in L^2(G)$ and suppose $u \in H^1(G)$ satisfies

$$a(u,v) = \int_G F\bar{v}\,dx \ , \qquad v \in H^1(G) \ . \tag{6.2}$$

Then $u \in H^2(G)$.

Proof: Let $\{(\varphi_k, G_k) : 1 \le k \le N\}$ be coordinate patches on ∂G and $\{\beta_k : 0 \le k \le N\}$ the partition-of-unity construction in Section II.2.3. Let B_k denote the support of β_k, $0 \le k \le N$. Since $u = \sum(\beta_k u)$ in G and each B_k is compact in $\mathbb{R}^n$, it is sufficient to show the following:

(a) $u|_{B_k \cap G} \in H^2(B_k \cap G)$, $1 \le k \le N$, and

(b) $\beta_0 u \in H^2(B_0)$.

The first case (a) will be proved below, and the second case (b) will follow from a straightforward modification of the first.

6.2

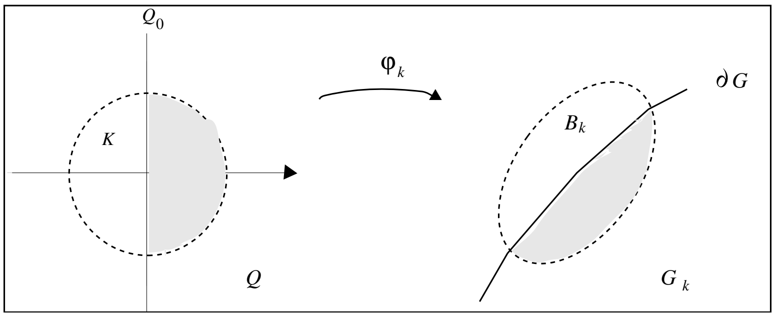
We fix k, $1 \le k \le N$, and note that the coordinate map $\varphi_k : Q \to G_k$ induces an isomorphism $\varphi_k^* : H^m(G_k \cap G) \to H^m(Q_+)$ for $m = 0, 1, 2$ by $\varphi_k^*(v) = v \circ \varphi_k$. Thus we define a continuous sesquilinear form on $H^1(Q_+)$ by

$$a^k\left(\varphi_k^*(w), \varphi_k^*(v)\right) \equiv \int_{G_k \cap G} \left\{ \sum_{i,j=1}^{n} a_{ij} \partial_j w \overline{\partial_j v} + \sum_{j=0}^{n} a_j \partial_j w \bar{v} \right\} dx \ . \qquad (6.3)$$

By making the appropriate change-of-variable in (6.3) and setting $w_k = \varphi_k^*(w)$, $v_k = \varphi_k^*(v)$, we obtain

$$a^k(w_k, v_k) = \int_{Q_+} \left\{ \sum_{i,j=1}^{n} a_{ij}^k \partial_i(w_k) \partial_j(v_k) + \sum_{j=0}^{n} a_j^k \partial_j(w_k) v_k \right\} dy \ . \qquad (6.4)$$

The resulting form (6.4) is strongly-elliptic on Q_+ (exercise).



Let u be the solution of (6.2) and let $v \in H^1(G \cap G_k)$ vanish in a neighborhood of ∂G_k. (That is, the support of v is contained in G_k.) Then the extension of v to all of G as zero on $G \sim G_k$ belongs to $H^1(G)$ and we obtain from (6.4) and (6.2)

$$a^k(\varphi_k^*(u), \varphi_k^*(v)) = a(u,v) = \int_{Q_+} F_k \varphi_k^*(v)\, dy \ ,$$

where $F_k \equiv \varphi_k^*(F) \cdot J(\varphi_k) \in L^2(Q_+)$. Letting $\mathcal{V}$ denote the space of those $v \in H^1(Q_+)$ which vanish in a neighborhood of ∂Q, and $u_k \equiv \varphi_k^*(u)$, we have shown that $u_k \in H^1(Q_+)$ satisfies

$$a^k(u_k, v_k) = \int_{Q_+} F_k v_k \, dy \ , \qquad v_k \in \mathcal{V} \tag{6.5}$$

where $a^k(\cdot,\cdot)$ is strongly elliptic with continuously differentiable coefficients with bounded derivatives and $F_k \in L^2(Q_+)$. We shall show that the restriction of u_k to the compact subset $K \equiv \varphi_k^{-1}(B_k)$ of Q belongs to $H^2(Q_+ \cap K)$. The first case (a) above will then follow.

6.3

Hereafter we drop the subscript "k" in (6.5). Thus, we have $u \in H^1(Q_+)$, $F \in L^2(Q_+)$ and

$$a(u,v) = \int_{Q_+} Fv \ , \qquad v \in \mathcal{V} \ . \tag{6.6}$$

Since $K \subset\subset Q$, there is by Lemma II.1.1 a $\varphi \in C_0^\infty(Q)$ such that $0 \le \varphi(x) \le 1$ for $x \in Q$ and $\varphi(x) = 1$ for $x \in K$. We shall first consider $\varphi \cdot u$.

Let w be a function defined on the half-space $\mathbb{R}_+^n$. For each $h \in \mathbb{R}$ we define a *translate* of w by

$$(\tau_h w)(x_1, x_2, \ldots, x_n) = w(x_1 + h, x_2, \ldots, x_n)$$

and a *difference* of w by

$$\nabla_h w = (\tau_h w - w)/h$$

if $h \ne 0$.

Lemma 6.2 *If $w, v \in L^2(Q_+)$ and the distance δ of the support of w to ∂Q is positive, then*

$$(\tau_h w, v)_{L^2(Q_+)} = (w, \tau_{-h} v)_{L^2(Q_+)}$$

for all $h \in \mathbb{R}$ with $|h| < \delta$.

Proof: This follows by the obvious change of variable and the observation that each of the above integrands is non-zero only on a compact subset of Q_+.

Corollary $\|\tau_h w\|_{L^2(Q_+)} = \|w\|_{L^2(Q_+)}$.

Lemma 6.3 *If* $w \in \mathcal{V}$, *then*

$$\|\nabla_h w\|_{L^2(Q_+)} \le \|\partial_1 w\|_{L^2(Q_+)} \ , \qquad 0 < |h| < \delta \ .$$

Proof: It follows from the preceding Corollary that it is sufficient to consider the case where $w \in C^1(\bar{G}) \cap \mathcal{V}$. Assuming this, and denoting the support of w by $\mathrm{supp}(w)$, we have

$$\nabla_h w(x) = h^{-1} \int_{x_1}^{x_1+h} \partial_1 w(t, x_2, \ldots, x_n)\, dt \ , \qquad w \in \mathrm{supp}(w) \ .$$

The Cauchy-Schwartz inequality gives

$$|\nabla_h w(x)| \le h^{-1/2} \left(\int_{x_1}^{x_1+h} |\partial_1 w(t, x_2, \ldots, x_n)|^2\, dt \right)^{1/2} \ , \qquad x \in \mathrm{supp}(w) \ ,$$

and this leads to

$$\begin{aligned}
\|\nabla_h w\|^2_{L^2(Q_+)} &\le h^{-1} \int_{\mathrm{supp}(w)} \int_{x_1}^{x_1+h} |\partial_1 w(t, x_2, \ldots, x_n)|^2\, dt\, dx \\
&= h^{-1} \int_{Q_+} \int_0^h |\partial_1 w(t + x_1, x_2, \ldots, x_n)|^2\, dt\, dx \\
&= h^{-1} \int_0^h \int_{Q_+} |\partial_1 w(t + x_1, \ldots, x_n)|^2\, dx\, dt \\
&= h^{-1} \int_0^h \int_{Q_+} |\partial_1 w(x_1, \ldots, x_n)|^2\, dx\, dt \\
&= \|\partial_1 w\|_{L^2(Q_+)} \ .
\end{aligned}$$

Corollary $\lim_{h \to 0}(\nabla_h w) = \partial_1 w$ *in* $L^2(Q_+)$.

Proof: $\{\nabla_h : 0 < |h| < \delta\}$ is a family of uniformly bounded operators on $L^2(Q_+)$, so it suffices to show the result holds on a dense subset.

We shall consider the forms

$$a_h(w,v) \equiv \int_{Q_+} \left\{ \sum_{i,j=1}^{n} (\nabla_h a_{ij}) \partial_i w \overline{\partial_j v} + \sum_{j=0}^{n} (\nabla_h a_j) \partial_j w \bar{v} \right\}$$

for $w, v \in \mathcal{V}$ and $|h| < \delta$, δ being given as in Lemma 6.2. Since the coefficients in (6.5) have bounded derivatives, the mean-value theorem shows

$$|a_h(w,v)| \le C\|w\|_{H^1(Q_+)} \cdot \|v\|_{H^1(Q_+)} \tag{6.7}$$

where the constant is independent of w, v and h. Finally, we note that for w, v and h as above

$$a(\nabla_h w, v) + a(w, \nabla_{-h} v) = -a_{-h}(\tau_{-h} w, v) \ . \tag{6.8}$$

This follows from a computation starting with the first term above and Lemma 6.2.

After this lengthy preparation we continue with the proof of Theorem 6.1. From (6.6) we have the identity

$$\begin{aligned} a(\nabla_h(\varphi u), v) = \ &\{a(\nabla_h(\varphi u), v) + a(\varphi u, \nabla_{-h} v)\} \\ &+ \{a(u, \varphi \nabla_{-h} v) - a(\varphi u, \nabla_{-h} v)\} - (F, \varphi \nabla_{-h} v)_{L^2(Q_+)} \end{aligned} \tag{6.9}$$

for $v \in \mathcal{V}$ and $0 < |h| < \delta$, δ being the distance from K to ∂Q. The first term can be bounded appropriately by using (6.7) and (6.8). The third is similarly bounded and so we consider the second term in (6.9). An easy computation gives

$$\begin{aligned} &a(u, \varphi \nabla_{-h} v) - a(\varphi u, \nabla_{-h} v) \\ &\quad = \int_{Q_+} \left\{ \sum_{i,j=1}^{n} a_{ij} (\partial_i u \partial_j \varphi \nabla_{-h} v - \partial_i \varphi u \nabla_{-h}(\partial_j v)) - \sum_{j=1}^{n} a_j \partial_j \varphi u (\nabla_{-h} v) \right\} . \end{aligned}$$

Thus, we obtain the estimate

$$|a(\nabla_h(\varphi u), v)| \le C\|v\|_{H^1(Q_+)} \ , \qquad v \in \mathcal{V} \ , \ 0 < |h| < \delta \ , \tag{6.10}$$

in which the constant C is independent of h and v. Since $a(\cdot,\cdot)$ is strongly-elliptic we may assume it is coercive (Exercise 6.2), so setting $v = \nabla_h(\varphi u)$ in (6.10) gives

$$c\|\nabla_h(\varphi u)\|^2_{H^1(Q_+)} \le C\|\nabla_h(\varphi u)\|_{H^1(Q_+)} \ , \qquad 0 < |h| < \delta \ , \tag{6.11}$$

hence, $\{\nabla_h(\varphi u) : |h| < \delta\}$ is bounded in the Hilbert space $H^1(Q_+)$. By Theorem I.6.2 there is a sequence $h_n \to 0$ for which $\nabla_{h_n}(\varphi u)$ converges weakly to some $w \in H^1(Q_+)$. But $\nabla_{h_n}(\varphi u)$ converges weakly in $L^1(Q_+)$ to $\partial_1(\varphi u)$, so the uniqueness of weak limits implies that $\partial_1(\varphi u) = w \in H^1(Q_+)$. It follows that $\partial_1^2(\varphi u) \in L^2(Q_+)$, and the same argument shows that each of the tangential derivatives $\partial_1^2 u, \partial_2^2 u, \ldots, \partial_{n-1}^2 u$ belongs to $L^2(K)$. (Recall $\varphi = 1$ on K.) This information together with the partial differential equation resulting from (6.6) implies that $a_{nn} \cdot \partial_n^2(u) \in L^2(K)$. The strong ellipticity implies a_{nn} has a positive lower bound on K, so $\partial_n^2 u \in L^2(K)$. Since n and all of its derivatives through second order are in $L^2(K)$, it follows from Theorem II.5.5 that $u \in H^2(K)$.

The preceding proves the case (a) above. The case (b) follows by using the differencing technique directly on $\beta_0 u$. In particular, we can compute differences on $\beta_0 u$ in *any* direction. The details are an easy modification of those of this section and we leave them as an exercise.

6.4

We discuss some extensions of Theorem 6.1. First, we note that the result and proof of Theorem 6.1 also hold if we replace $H^1(G)$ by $H_0^1(G)$. This results from the observation that the subspace $H_0^1(G)$ is invariant under multiplication by smooth functions and translations and differences in tangential directions along the boundary of G. Thus we obtain a regularity result for the Dirichlet problem.

Theorem 6.4 *Let $u \in H_0^1(G)$ satisfy*

$$a(u, v) = \int_G F\bar{v} \ , \qquad v \in H_0^1(G)$$

where the set $G \subset \mathbb{R}^n$ and sesquilinear form $a(\cdot,\cdot)$ are given as in Theorem 6.1, and $F \in L^2(G)$. Then $u \in H^2(G)$.

When the data in the problem is smoother yet, one expects the same to be true of the solution. The following describes the situation which is typical of second-order elliptic boundary value problems.

Definition. Let V be a closed subspace of $H^1(G)$ with $H_0^1(G) \leq V$, and let $a(\cdot,\cdot)$ be a continuous sesquilinear form on V. Then $a(\cdot,\cdot)$ is called *k-regular on V* if for every $F \in H^s(G)$ with $0 \leq s \leq k$ and every solution $u \in V$ of

$$a(u, v) = (F, v)_{L^2(G)} \ , \qquad v \in V$$

we have $u \in H^{2+s}(G)$.

Theorems 6.1 and 6.4 give sufficient conditions for the form $a(\cdot,\cdot)$ given by (6.1) to be 0-regular over $H^1(G)$ and $H_0^1(G)$, respectively. Moreover, we have the following.

Theorem 6.5 *The form $a(\cdot,\cdot)$ given by* (6.1) *is k-regular over $H^1(G)$ and $H_0^1(G)$ if ∂G is a C^{2+k}-manifold and the coefficients $\{a_{ij}, a_j\}$ all belong to $C^{1+k}(\bar{G})$.*

7 Closed operators, adjoints and eigenfunction expansions

7.1

We were led in Section 2 to consider a linear map $A : D \to H$ whose domain D is a subspace of the Hilbert space H. We shall call such a map an (*unbounded*) *operator* on the Hilbert space H. Although an operator is frequently not continuous (with respect to the H-norm on D) it may have the property we now consider. The *graph* of A is the subspace

$$G(A) = \{[x, Ax] : x \in D\}$$

of the product $H \times H$. (This product is a Hilbert space with the scalar product

$$([x_1, x_2], [y_1, y_2])_{H\times H} = (x_1, y_1)_H + (x_2, y_2)_H \ .$$

The addition and scalar multiplication are defined componentwise.) The operator A on H is called *closed* if $G(A)$ is a closed subset of $H \times H$. That is, A is closed if for any sequence $x_n \in D$ such that $x_n \to x$ and $Ax_n \to y$ in H, we have $x \in D$ and $Ax = y$.

Lemma 7.1 *If A is closed and continuous (i.e., $\|Ax\|_H \le K\|x\|_H$, $x \in H$) then D is closed.*

Proof: If $x_n \in D$ and $x_n \to x \in H$, then $\{x_n\}$ and, hence, $\{Ax_n\}$ are Cauchy sequences. H is complete, so $Ax_n \to y \in H$ and $G(A)$ being closed implies $x \in D$.

When D is dense in H we define the *adjoint* of A as follows. The domain of the operator A^* is the subspace D^* of all $y \in H$ such that the map

$x \mapsto (Ax, y)_H : D \to \mathbb{K}$ is continuous. Since D is dense in H, Theorem I.4.5 asserts that for each such $y \in D^*$ there is a unique $A^*y \in H$ such that

$$(Ax, y) = (x, A^*y) \ , \qquad x \in D \ , \ y \in D^* \ . \tag{7.1}$$

Then the function $A^* : D^* \to H$ is clearly linear and is called the *adjoint* of A. The following is immediate from (7.1).

Lemma 7.2 *A^* is closed.*

Lemma 7.3 *If $D = H$, then A^* is continuous, hence, D^* is closed.*

Proof: If A^* is not continuous there is a sequence $x_n \in D^*$ such that $\|x_n\| = 1$ and $\|A^*x_n\| \to \infty$. From (7.1) it follows that for each $x \in H$,

$$|(x, A^*x_n)_H| = |(Ax, x_n)_H| \leq \|Ax\|_H \ ,$$

so the sequence $\{A^*x_n\}$ is weakly bounded. But Theorem I.6.1 implies that it is bounded, a contradiction.

Lemma 7.4 *If A is closed, then D^* is dense in H.*

Proof: Let $y \in H$, $y \neq 0$. Then $[0, y] \notin G(A)$ and $G(A)$ closed in $H \times H$ imply there is an $f \in (H \times H)'$ such that $f[G(A)] = \{0\}$ and $f(0, y) \neq 0$. In particular, let $P : H \times H \to G(A)^\perp$ be the projection onto the orthogonal complement of $G(A)$ in $H \times H$, define $[u, v] = P[0, y]$, and set

$$f(x_1, x_2) = (u, x_1)_H + (v, x_2)_H \ , \qquad x_1, x_2 \in H \ .$$

Then we have

$$0 = f(x, Ax) = (u, x)_H + (v, Ax)_H \ , \qquad x \in D$$

so $v \in D^*$, and $0 \neq f(0, y) = (v, y)_H$. The above shows $(D^*)^\perp = \{0\}$, so D^* is dense in H.

The following result is known as the *closed-graph theorem*.

Theorem 7.5 *Let A be an operator on H with domain D. Then A is closed and $D = H$ if and only if $A \in \mathcal{L}(H)$.*

Proof: If A is closed and $D = H$, then Lemma 7.3 and Lemma 7.4 imply $A^* \in \mathcal{L}(H)$. Then Theorem I.5.2 shows $(A^*)^* \in \mathcal{L}(H)$. But (7.1) shows $A = (A^*)^*$, so $A \in \mathcal{L}(H)$. The converse in immediate.

The operators with which we are most often concerned are adjoints of another operator. The preceding discussion shows that the domain of such an operator, i.e., an adjoint, is all of H if and only if the operator is continuous. Thus, we shall most often encounter unbounded operators which are closed and densely defined.

We give some examples in $H = L^2(G)$, $G = (0, 1)$.

7.2

Let $D = H_0^1(G)$ and $A = i\partial$. If $[u_n, Au_n] \in G(A)$ converges to $[u, v]$ in $H \times H$, then in the identity

$$\int_0^1 Au_n \varphi \, dx = -i \int_0^1 u_n \, D\varphi \, dx \ , \qquad \varphi \in C_0^\infty(G) \ ,$$

we let $n \to \infty$ and thereby obtain

$$\int_0^1 v\varphi \, dx = -i \int_0^1 u \, D\varphi \, dx \ , \qquad \varphi \in C_0^\infty(G) \ .$$

This means $v = i\partial u = Au$ and $u_n \to u$ in $H^1(G)$. Hence $u \in H_0^1(G)$, and we have shown A is closed.

To compute the adjoint, we note that

$$\int_0^1 Au\bar{v} \, dx = \int_0^1 u\bar{f} \, dx \ , \qquad u \in H_0^1(G)$$

for some pair $v, f \in L^2(G)$ if and only if $v \in H^1(G)$ and $f = i\partial v$. Thus $D^* = H^1(G)$ and $A^* = i\partial$ is a proper extension of A.

7.3

We consider the operator A^* above: on its domain $D^* = H^1(G)$ it is given by $A^* = i\partial$. Since A^* is an adjoint it is closed. We shall compute $A^{**} = (A^*)^*$, the second adjoint of A. We first note that the pair $[u, f] \in H \times H$ is in the graph of A^{**} if and only if

$$\int_0^1 A^* v\bar{u} \, dx = \int_0^1 v\bar{f} \, dx \ , \qquad v \in H^1(G) \ .$$

This holds for all $v \in C_0^\infty(G)$, so we obtain $i\partial u = f$. Substituting this into the above and using Theorem II.1.6, we obtain

$$i\int_0^1 \partial(v\bar{u})\,dx = \int_0^1 \Big[(i\partial v)\bar{u} - v\overline{(i\partial u)}\Big]\,dx = 0\ ,$$

hence, $v(1)\bar{u}(1) - v(0)\bar{u}(0) = 0$ for all $v \in H^1(G)$. But this implies $u(0) = u(1) = 0$, hence, $u \in H_0^1(G)$. From the above it follows that $A^{**} = A$.

7.4

Consider the operator $B = i\partial$ on $L^2(G)$ with domain $D(B) = \{u \in H^1(G) : u(0) = cu(1)\}$ where $c \in \mathbb{C}$ is given. If $v, f \in L^2(G)$, then $B^*v = f$ if and only if

$$\int_0^1 i\partial u \cdot \bar{v}\,dx = \int_0^1 u\bar{f}\,dx\ , \qquad u \in D\ .$$

But $C_0^\infty(G) \le D$ implies $v \in H^1(G)$ and $i\partial v = f$. We substitute this identity in the above and obtain

$$0 = i\int_0^1 \partial(u\bar{v}\,)\,dx = iu(1)[\bar{v}(1) - c\bar{v}(0)]\ , \qquad u \in D\ .$$

The preceding shows that $v \in D(B^*)$ only if $v \in H^1(0,1)$ and $v(1) = \bar{c}v(0)$. It is easy to show that every such v belongs to $D(B^*)$, so we have shown that $D(B^*) = \{v \in H^1(G) : v(1) = \bar{c}v(0)\}$ and $B^* = i\partial$.

7.5

We return to the situation of Section 2.2. Let $a(\cdot,\cdot)$ be a continuous sesquilinear form on the Hilbert space V which is dense and continuously imbedded in the Hilbert space H. We let D be the set of all $u \in V$ such that the map $v \mapsto a(u,v)$ is continuous on V with the norm of H. For such a $u \in D$, there is a unique $Au \in H$ such that

$$a(u,v) = (Au,v)_H\ , \qquad u \in D\ ,\ v \in V\ . \tag{7.2}$$

This defines a linear operator A on H with domain D.

Consider the (adjoint) sesquilinear form on V defined by $b(u,v) = \overline{a(v,u)}$, $u, v \in V$. This gives another operator B on H with domain $D(B)$ determined as before by

$$b(u,v) = (Bu,v)_H\ , \qquad u \in D(B)\ ,\ v \in V\ .$$

Theorem 7.6 *Assume there is a* $\lambda > 0$ *and* $c > 0$ *such that*

$$\mathrm{Re}\ a(u,u) + \lambda|u|_H^2 \geq c\|u\|_V^2\ , \qquad u \in V\ . \tag{7.3}$$

Then D *is dense in* H, A *is closed, and* $A^* = B$, *hence,* $D^* = D(B)$.

Proof: Theorem 2.2 shows D is dense in H. If we prove $A^* = B$, then by symmetry we obtain $B^* = A$, hence A is closed by Lemma 7.2.

Suppose $v \in D(B)$. Then for all $u \in D(A)$ we have $(Au,v)_H = a(u,v) = \overline{b(v,u)} = \overline{(Bv,u)_H}$, hence, $(Au,v)_H = (u,Bv)_H$. This shows $D(B) \leq D^*$ and $A^*|_{D(B)} = B$. We need only to verify that $D(B) = D^*$. Let $u \in D^*$. Since $B + \lambda$ is surjective, there is a $u_0 \in D(B)$ such that $(B+\lambda)u_0 = (A^* + \lambda)u$. Then for all $v \in D$ we have

$$\begin{aligned} ((A+\lambda)v,u)_H &= (v,(B+\lambda)u_0)_H = a(v,u_0) + \lambda(v,u_0)_H \\ &= ((A+\lambda)v,u_0)_H\ . \end{aligned}$$

But $A+\lambda$ is a surjection, so this implies $u = u_0 \in D(B)$. Hence, $D^* = D(B)$.

For those operators as above which arise from a symmetric sesquilinear form on a space V which is compactly imbedded in H, we can apply the eigenfunction expansion theory for self-adjoint compact operators.

Theorem 7.7 *Let* V *and* H *be Hilbert spaces with* V *dense in* H *and assume the injection* $V \hookrightarrow H$ *is compact. Let* $A : D \to H$ *be the linear operator determined as above by a continuous sesquilinear form* $a(\cdot,\cdot)$ *on* V *which we assume is* V*-elliptic and symmetric:*

$$a(u,v) = \overline{a(v,u)}\ , \qquad u,v \in V\ .$$

Then there is a sequence $\{v_j\}$ *of eigenfunctions of* A *with*

$$\left.\begin{array}{c} Av_j = \lambda_j v_j\ , \qquad |v_j|_H = 1\ , \\ (v_i,v_j)_H = 0\ , \qquad i \neq j\ , \\ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to +\infty \quad \text{as } n \to +\infty\ , \end{array}\right\} \tag{7.4}$$

and $\{v_j\}$ *is a basis for* H.

Proof: From Theorem 7.6 it follows that $A = A^*$ and, hence, $A^{-1} \in \mathcal{L}(H)$ is self-adjoint. The V-elliptic condition (5.5) shows that $A^{-1} \in \mathcal{L}(H,V)$.

Since the injection $V \hookrightarrow H$ is compact, it follows that $A^{-1} : H \to V \to H$ is compact. We apply Theorem I.7.5 to obtain a sequence $\{v_j\}$ of eigenfunctions of A^{-1} which are orthonormal in H and form a basis for $D = \mathrm{Rg}(A^{-1})$. If their corresponding eigenvalues are denoted by $\{\mu_j\}$, then the symmetry of $a(\cdot,\cdot)$ and (5.5) shows that each μ_j is positive. We obtain (7.4) by setting $\lambda_j = 1/\mu_j$ for $j \geq 1$ and noting that $\lim_{j\to\infty} \mu_j = 0$.

It remains to show $\{v_j\}$ is a basis for H. (We only know that it is a basis for D.) Let $f \in H$ and $u \in D$ with $Au = f$. Let $\sum b_j v_j$ be the Fourier series for f, $\sum c_j v_j$ the Fourier series for u, and denote their respective partial sums by

$$u_n = \sum_{j=1}^{n} c_j v_j \quad , \quad f_n = \sum_{j=1}^{n} b_j v_j \ .$$

We know $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} f_n = f_\infty$ exists in H (cf. Exercise I.7.2). For each $j \geq 1$ we have

$$b_j = (Au, v_j)_H = (u, Av_j)_H = \lambda_j c_j \ ,$$

so $Au_n = f_n$ for all $n \geq 1$. Since A is closed, it follows $Au = f_\infty$, hence, $f = \lim_{n\to\infty} f_n$ as was desired.

If we replace A by $A + \lambda$ in the proof of Theorem 7.7, we observe that ellipticity of $a(\cdot,\cdot)$ is not necessary but only that $a(\cdot,\cdot)+\lambda(\cdot,\cdot)_H$ be V-elliptic for some $\lambda \in \mathbb{R}$.

Corollary 7.8 *Let V and H be given as in Theorem 7.7, let $a(\cdot,\cdot)$ be continuous, sesquilinear, and symmetric. Assume also that*

$$a(v,v) + \lambda |v|_H^2 \geq c\|v\|_V^2 \ , \qquad v \in V$$

for some $\lambda \in \mathbb{R}$ and $c > 0$. Then there is an orthonormal sequence of eigenfunctions of A which is a basis for H and the corresponding eigenvalues satisfy $-\lambda < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to +\infty$ as $n \to +\infty$.

We give some examples in $H = L^2(G)$, $G = (0,1)$. These eigenvalue problems are known as *Sturm-Liouville problems*. Additional examples are described in the exercises.

7.6

Let $V = H_0^1(G)$ and define $a(u,v) = \int_0^1 \partial u \overline{\partial v}\, dx$. The compactness of $V \to H$ follows from Theorem II.5.7 and Theorem 5.3 shows $a(\cdot,\cdot)$ is $H_0^1(G)$-elliptic. Thus Theorem 7.7 holds; it is a straightforward exercise to compute the eigenfunctions and corresponding eigenvalues for the operator $A = -\partial^2$ with domain $D(A) = H_0^1(G) \cap H^2(G)$:

$$v_j(x) = 2\sin(j\pi x) \ , \quad \lambda = (j\pi)^2 \ , \qquad j = 1, 2, 3, \dots \ .$$

Since $\{v_j\}$ is a basis for $L^2(G)$, each $F \in L^2(G)$ has a Fourier sine-series expansion. Similar results hold in higher dimension for, e.g., the eigenvalue problem

$$\begin{cases} -\Delta_n v(x) = \lambda v(x) \ , & x \in G \ , \\ v(s) = 0 \ , & s \in \partial G \ , \end{cases}$$

but the actual computation of the eigenfunctions and eigenvalues is difficult except for very special regions $G \subset \mathbb{R}^n$.

7.7

Let $V = H^1(G)$ and choose $a(\cdot,\cdot)$ as above. The compactness follows from Theorem II.5.8 so Corollary 7.8 applies for any $\lambda > 0$ to give a basis of eigenfunctions for $A = -\partial^2$ with domain $D(A) = \{v \in H^2(G) : v'(0) = v'(1) = 0\}$:

$$\begin{aligned} v_0(x) &= 1 \ , \quad v_j(x) = 2\cos(j\pi x) \ , \ j \geq 1 \ , \\ \lambda_j &= (j\pi)^2 \ , \qquad\qquad\qquad\quad j \geq 0 \ . \end{aligned}$$

As before, similar results hold for the Laplacean with boundary conditions of second type in higher dimensions.

7.8

Let $a(\cdot,\cdot)$ be given as above but set $V = \{v \in H^1(G) : v(0) = v(1)\}$. Then we can apply Corollary 7.8 to the *periodic eigenvalue problem* (cf. (4.5))

$$\begin{aligned} -\partial^2 v(x) &= \lambda v(x) \ , \qquad 0 < x < 1 \ , \\ v(0) &= v(1) \ , \quad v'(0) = v'(1) \ . \end{aligned}$$

The eigenfunction expansion is just the standard Fourier series.

Exercises

1.1. Use Theorem 1.1 to show the problem $-\Delta_n u = F$ in G, $u = 0$ on ∂G is well-posed. Hint: Use Theorem II.2.4 to obtain an appropriate norm on $H_0^1(G)$.

1.2. Use Theorem 1.1 to solve (1.1) with the boundary condition $\partial u/\partial\nu + u = 0$ on ∂G. Hint: Use $(u,v)_V \equiv (u,v)_{H^1(G)} + (\gamma u, \gamma v)_{L^2(\partial G)}$ on $H^1(G)$.

2.1. Give the details of the construction of α, β in (2.2).

2.2. Verify the remark on $H = L^2(G)$ following (2.5) (cf. Section I.5.3).

2.3. Use Theorem I.1.1 to construct the F which appears after (2.6). Check that it is continuous.

2.4. Show that $a(u,v) = \int_0^1 \partial u(x)\partial\bar{v}(x)\,dx$, $V = \{u \in H^1(0,1) : u(0) = 0\}$, and $f(v) \equiv v(1/2)$ are admissible data in Theorem 2.1. Find a formula for the unique solution of the problem.

2.5. In Theorem 2.1 the continuous dependence of the solution u on the data f follows from the estimate made in the theorem. Consider the two abstract boundary value problems $\mathcal{A}_1 u_1 = f$ and $\mathcal{A}_2 u_2 = f$ where $f \in V'$, and $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{L}(V, V')$ are coercive with constants c_1, c_2, respectively. Show that the following estimates holds:

$$\|u_1 - u_2\| \le (1/c_1)\|(\mathcal{A}_2 - \mathcal{A}_1)u_2\| \ ,$$
$$\|u_1 - u_2\| \le (1/c_1 c_2)\|\mathcal{A}_2 - \mathcal{A}_1\|\,\|f\| \ .$$

Explain how these estimates show that the solution of (2.1) depends continuously on the form $a(\cdot,\cdot)$ or operator $\mathcal{A}$.

3.1. Show (3.3) implies (3.1) in Theorem 3.1.

3.2. (Non-homogeneous Boundary Conditions.) In the situation of Theorem 3.1, assume we have a closed subspace V_1 with $V_0 \subset V_1 \subset V$ and $u_0 \in V$. Consider the problem to find

$$u \in V \ , \quad u - u_0 \in V_1 \ , \quad a(u,v) = f(v) \ , \quad v \in V_1 \ .$$

(a) Show this problem is well-posed if $a(\cdot,\cdot)$ is V_1-coercive.

(b) Characterize the solution by $u - u_0 \in V_1$, $u \in D_0$, $Au = F$, and $\partial u(v) + a_2(\gamma u, \gamma v) = g(\gamma v)$, $v \in V_1$.

(c) Construct an example of the above with $V_0 = H_0^1(G)$, $V = H^1(G)$, $V_1 = \{v \in V : v|_\Gamma = 0\}$, where $\Gamma \subset \partial G$ is given.

4.1. Verify that the formal operator and Green's theorem are as indicated in Section 4.1.

4.2. Characterize the boundary value problem resulting from the choice of $V = \{v \in H^1(G) : v = \text{const. on } G_0\}$ in Section 4.2, where $G_0 \subset G$ is given.

4.3. When G is a cube in $\mathbb{R}^n$, show (4.5) is related to a problem on $\mathbb{R}^n$ with *periodic* solutions.

4.4. Choose V in Section 4.2 so that the solution $u : \mathbb{R}^n \to \mathbb{K}$ is periodic in each coordinate direction.

5.1. Formulate and solve the problem (4.8) with non-homogeneous data prescribed on ∂G and Σ.

5.2. Find choices for V in Section 4.3 which lead to well-posed problems. Characterize the solution by a boundary value problem.

5.3. Prove Corollary 5.4.

5.4. Discuss coercivity of the form (4.6). Hint: $\mathrm{Re}(\int_{\partial G} \frac{\partial u}{\partial \tau} \bar{u}\, ds) = 0$.

6.1. Show (6.4) is strongly-elliptic on Q_+.

6.2. Show that the result of Theorem 6.1 holds for $a(\cdot,\cdot)$ if and only if it holds for $a(\cdot,\cdot) + \lambda(\cdot,\cdot)_{L^2(G)}$. Hence, one may infer coercivity from strong ellipticity without loss of generality.

6.3. If $u \in H^1(G)$, show $\nabla_h(u)$ converges weakly in $L^2(G)$ to $\partial_1(u)$.

6.4. Prove the case (b) in Theorem 6.1.

6.5. Prove Theorem 6.5.

6.6. Give sufficient conditions for the solution of (6.2) to be a classical solution in $C_u^2(G)$.

7.1. Prove Lemma 7.2 of Section 7.1.

7.2. Compute the adjoint of $\partial : \{v \in H^1(G) : v(0) = 0\} \to L^2(G)$, $G = (0,1)$.

7.3. Let $D \leq H^2(G)$, $G = (0,1)$, $a_1(\cdot), a_2(\cdot) \in C^1(\bar{G})$, and define $L : D \to L^2(G)$ by $Lu = \partial^2 u + a_1 \partial u + a_2 u$. The *formal adjoint* of L is defined by

$$L^* v(\varphi) = \int_0^1 v(x)\overline{L\varphi(x)}\, dx \ , \qquad v \in L^2(G) \ , \ \varphi \in C_0^\infty(G) \ .$$

(a) Show $L^* v = \partial^2 v - \partial(\bar{a}_1 v) + \bar{a}_2 v$ in $\mathcal{D}^*(G)$.

(b) If $u, v \in H^2(G)$, then $\int_0^1 (Lu\bar{v} - uL^*\bar{v}\,)\, dx = J(u,v)|_{x=0}^{x=1}$, where $J(u,v) = \bar{v}\partial u - u\partial\bar{v} + a_1 u\bar{v}$.

(c) $D(L^*) = \{v \in H^2(G) : J(u,v)|_{x=0}^{x=1} = 0, \text{ all } u \in D\}$ determines the domain of the $L^2(G)$-adjoint.

(d) Compute $D(L^*)$ when $L = \partial^2 + 1$ and each of the following:

(i) $D = \{u : u(0) = u'(0) = 0\}$,

(ii) $D = \{u : u(0) = u(1) = 0\}$,

(iii) $D = \{u : u(0) = u(1),\ u'(0) = u'(1)\}$.

7.4. Let A be determined by $\{a(\cdot,\cdot), V, H\}$ as in (7.2) and A_λ by $\{a(\cdot,\cdot) + \lambda(\cdot,\cdot)_H, V, H\}$. Show $D(A_\lambda) = D(A)$ and $A_\lambda = A + \lambda I$.

7.5. Let H_j, V_j be Hilbert spaces with V_j continuously embedded in H_j for $j = 1, 2$. Show that if $T \in \mathcal{L}(H_1, H_2)$ and if $T_1 \equiv T|_{V_1} \in L(V_1, V_2)$, then $T_1 \in \mathcal{L}(V_1, V_2)$.

7.6. In the situation of Section 6.4, let $a(\cdot,\cdot)$ be 0-regular on V and assume $a(\cdot,\cdot)$ is also V-elliptic. Let A be determined by $\{a(\cdot,\cdot), V, L^2(G)\}$ as in (7.2).

(a) Show $A^{-1} \in \mathcal{L}(L^2(G), V)$.

(b) Show $A^{-1} \in \mathcal{L}(L^2(G), H^2(G))$.

(c) If $a(\cdot,\cdot)$ is k-regular, show $A^{-p} \in \mathcal{L}(L^2(G), H^{2+k}(G))$ if p is sufficiently large.

7.7. Let A be *self-adjoint* on the complex Hilbert space H. That is, $A = A^*$.

(a) Show that if $\mathrm{Im}(\lambda) \neq 0$, then $\lambda - A$ is invertible and $|\,\mathrm{Im}(\lambda)|\, \|x\|_H \leq \|(\lambda - A)x\|_H$ for all $x \in D(A)$.

(b) $\mathrm{Rg}(\lambda - A)$ is dense in H.

(c) Show $(\lambda - A)^{-1} \in \mathcal{L}(H)$ and $\|(\lambda - A)^{-1}\| \leq |\operatorname{Im}(\lambda)|^{-1}$.

7.8. Show Theorem 7.7 applies to the mixed Dirichlet-Neumann eigenvalue problem

$$-\partial^2 v = \lambda v(x) \ , \qquad 0 < x < 1 \ , \ v(0) = v'(1) = 0 \ .$$

Compute the eigenfunctions.

7.9. Show Corollary 7.8 applies to the eigenvalue problem with boundary conditions of third type

$$-\partial^2 v(x) = \lambda v(x) \ , \qquad 0 < x < 1 \ ,$$
$$\partial v(0) - hv(0) = 0 \quad , \quad \partial v(1) + hv(1) = 0 \ ,$$

where $h > 0$. Compute the eigenfunctions.

7.10. Take $c\bar{c} = 1$ in Section 7.4 and discuss the eigenvalue problem $Bv = \lambda v$.

7.11. In the proof of Theorem 7.7, deduce that $\{v_j\}$ is a basis for H directly from the fact that $\bar{D} = H$.

Chapter IV

First Order Evolution Equations

1 Introduction

We consider first an initial-boundary value problem for the equation of heat conduction. That is, we seek a function $u : [0, \pi] \times [0, \infty] \to \mathbb{R}$ which satisfies the partial differential equation

$$u_t = u_{xx} \ , \qquad 0 < x < \pi \ , \ t > 0 \tag{1.1}$$

with the boundary conditions

$$u(0,t) = 0 \ , \quad u(\pi, t) = 0 \ , \qquad t > 0 \tag{1.2}$$

and the initial condition

$$u(x, 0) = u_0(x) \ , \qquad 0 < x < \pi \ . \tag{1.3}$$

A standard technique for solving this problem is the method of separation of variables. One begins by looking for non-identically-zero solutions of (1.1) of the form

$$u(x,t) = v(x)T(t)$$

and is led to consider the pair of ordinary differential equations

$$v'' + \lambda v = 0 \quad , \quad T' + \lambda T = 0$$

and the boundary conditions $v(0) = v(\pi) = 0$. This is an eigenvalue problem for $v(x)$ and the solutions are given by $v_n(x) = \sin(nx)$ with corresponding eigenvalues $\lambda_n = n^2$ for integer $n \geq 1$ (cf. Section II.7.6).

The second of the pair of equations has corresponding solutions

$$T_n(t) = e^{-n^2 t}$$

and we thus obtain a countable set

$$u_n(x,t) = e^{-n^2 t} \sin(nx)$$

of functions which satisfy (1.1) and (1.2). The solution of (1.1), (1.2) and (1.3) is then obtained as the series

$$u(x,t) = \sum_{n=1}^{\infty} u_0^n e^{-n^2 t} \sin(nx) \tag{1.4}$$

where the $\{u_0^n\}$ are the Fourier coefficients

$$u_0^n = \frac{2}{\pi} \int_0^\pi u_0(x) \sin(nx)\, dx\ , \qquad n \geq 1\ ,$$

of the initial function $u_0(x)$.

We can regard the representation (1.4) of the solution as a function $t \mapsto S(t)$ from the non-negative reals $\mathbb{R}_0^+$ to the bounded linear operators on $L^2[0,\pi]$. We define $S(t)$ to be the operator given by

$$S(t)u_0(x) = u(x,t)\ ,$$

so $S(t)$ assigns to each function $u_0 \in L^2[0,\pi]$ that function $u(\cdot,t) \in L^2[0,\pi]$ given by (1.4). If $t_1, t_2 \in \mathbb{R}_0^+$, then we obtain for each $u_0 \in L^2[0,\pi]$ the equalities

$$\begin{aligned}
S(t_1)u_0(x) &= \sum_{n=1}^{\infty} (u_0^n e^{-n^2 t_1}) \sin(nx) \\
S(t_2)S(t_1)u_0(x) &= \sum_{n=1}^{\infty} (u_0^n e^{-n^2 t_1}) \sin(nx) e^{-n^2 t_2} \\
&= \sum_{n=1}^{\infty} u_0^n \sin(nx) e^{-n^2 (t_1+t_2)} \\
&= S(t_1+t_2)u_0(x)\ .
\end{aligned}$$

Since u_0 is arbitrary, this shows that

$$S(t_1) \cdot S(t_2) = S(t_1 + t_2) \ , \qquad t_1, t_2 \geq 0 \ .$$

This is the *semigroup identity*. We can also show that $S(0) = I$, the identity operator, and that for each u_0, $S(t)u_0 \to u_0$ in $L^2[0,\pi]$ as $t \to 0^+$. Finally, we find that each $S(t)$ has norm $\leq e^{-t}$ in $\mathcal{L}(L^2[0,\pi])$. The properties of $\{S(t) : t \geq 0\}$ that we have obtained here will go into the definition of contraction semigroups. We shall find that each contraction semigroup is characterized by a representation for the solution of a corresponding Cauchy problem.

Finally we show how the semigroup $\{S(t) : t \geq 0\}$ leads to a representation of the solution of the non-homogeneous partial differential equation

$$u_t = u_{xx} + f(x,t) \ , \qquad 0 < x < \pi \ , \ t > 0 \tag{1.5}$$

with the boundary conditions (1.2) and initial condition (1.3). Suppose that for each $t > 0$, $f(\cdot, t) \in L^2[0,\pi]$ and, hence, has the eigenfunction expansion

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin(nx) \quad , \quad f_n(t) \equiv \frac{2}{\pi} \int_0^{\pi} f(\xi, t) \sin(n\xi) \, d\xi \ . \tag{1.6}$$

We look for the solution in the form $u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin(nx)$ and find from (1.5) and (1.3) that the coefficients must satisfy

$$\begin{aligned} &u_n'(t) + n^2 u_n(t) = f_n(t) \ , \ t \geq 0 \ , \\ &u_n(0) = u_n^0 \ , \qquad\qquad\qquad n \geq 1 \ . \end{aligned}$$

Hence we have

$$u_n(t) = u_n^0 e^{-n^2 t} + \int_0^t e^{-n^2(t-\tau)} f_n(\tau) \, d\tau$$

and the solution is given by

$$u(x,t) = S(t)u_0(x) + \int_0^t \int_0^{\pi} \left\{ \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2(t-\tau)} \sin(nx) \sin(n\xi) \right\} f(\xi, \tau) \, d\xi \, d\tau \ .$$

But from (1.6) it follows that we have the representation

$$u(\cdot, t) = S(t)u_0(\cdot) + \int_0^t S(t-\tau) f(\cdot, \tau) \, d\tau \tag{1.7}$$

for the solution of (1.5), (1.2), (1.3). The preceding computations will be made precise in this chapter and (1.7) will be used to prove existence and uniqueness of a solution.

2 The Cauchy Problem

Let H be a Hilbert space, $D(A)$ a subspace of H, and $A \in L(D(A), H)$. We shall consider the evolution equation

$$u'(t) + Au(t) = 0 \ . \tag{2.1}$$

The *Cauchy problem* is to find a function $u \in C([0,\infty], H) \cap C^1((0,\infty), H)$ such that, for $t > 0$, $u(t) \in D(A)$ and (2.1) holds, and $u(0) = u_0$, where the initial value $u_0 \in H$ is prescribed.

Assume that for every $u_0 \in D(A)$ there exists a unique solution of the Cauchy problem. Define $S(t)u_0 = u(t)$ for $t \geq 0$, $u_0 \in D(A)$, where $u(\cdot)$ denotes that solution of (2.1) with $u(0) = u_0$. If $u_0, v_0 \in D(A)$ and if $a, b \in \mathbb{R}$, then the function $t \mapsto aS(t)u_0 + bS(t)v_0$ is a solution of (2.1), since A is linear, and the uniqueness of solutions then implies

$$S(t)(au_0 + bv_0) = aS(t)u_0 + bS(t)v_0 \ .$$

Thus, $S(t) \in L(D(A))$ for all $t \geq 0$. If $u_0 \in D(A)$ and $\tau \geq 0$, then the function $t \mapsto S(t+\tau)u_0$ satisfies (2.1) and takes the initial value $S(\tau)u_0$. The uniqueness of solutions implies that

$$S(t+\tau)u_0 = S(t)S(\tau)u_0 \ , \qquad u_0 \in D(A) \ .$$

Clearly, $S(0) = I$.

We define the operator A to be *accretive* if

$$\mathrm{Re}(Ax, x)_H \geq 0 \ , \qquad x \in D(A) \ .$$

If A is accretive and if u is a solution of the Cauchy problem for (2.1), then

$$\begin{aligned} D_t(\|u(t)\|^2) &= 2\,\mathrm{Re}(u'(t), u(t))H \\ &= -2\,\mathrm{Re}(Au(t), u(t))_H \leq 0 \ , \qquad t > 0 \ , \end{aligned}$$

so it follows that $\|u(t)\| \leq \|u(0)\|$, $t \geq 0$. This shows that

$$\|S(t)u_0\| \leq \|u_0\| \ , \qquad u_0 \in D(A) \ , \ t \geq 0 \ ,$$

so each $S(t)$ is a contraction in the H-norm and hence has a unique extension to the closure of $D(A)$. When $D(A)$ is dense, we thereby obtain a contraction semigroup on H.

Definition. A *contraction semigroup* on H is a set $\{S(t) : t \geq 0\}$ of linear operators on H which are contractions and satisfy

$$S(t+\tau) = S(t) \cdot S(\tau) \ , \quad S(0) = I \ , \ t, \tau \geq 0 \ , \tag{2.2}$$

$$S(\cdot)x \in C([0,\infty), H) \ , \qquad x \in H \ . \tag{2.3}$$

The *generator* of the contraction semigroup $\{S(t) : t \geq 0\}$ is the operator with domain

$$D(B) = \left\{ x \in H : \lim_{h \to 0^+} h^{-1}(S(h) - I)x = D^+(S(0)x) \text{ exists in } H \right\}$$

and value $Bx = \lim_{h \to 0^+} h^{-1}(S(h) - I)x = D^+(S(0)x)$. Note that Bx is the right-derivative at 0 of $S(t)x$.

The equation (2.2) is the *semigroup identity*. The definition of solution for the Cauchy problem shows that (2.3) holds for $x \in D(A)$, and an elementary argument using the uniform boundedness of the (contraction) operators $\{S(t) : t \geq 0\}$ shows that (2.3) holds for all $x \in H$. The property (2.3) is the *strong continuity* of the semigroup.

Theorem 2.1 *Let $A \in L(D(A), H)$ be accretive with $D(A)$ dense in H. Suppose that for every $u_0 \in D(A)$ there is a unique solution $u \in C^1([0,\infty), H)$ of* (2.1) *with $u(0) = u_0$. Then the family of operators $\{S(t) : t \geq 0\}$ defined as above is a contraction semigroup on H whose generator is an extension of $-A$.*

Proof: Note that uniqueness of solutions is implied by A being accretive, so the semigroup is defined as above. We need only to verify that $-A$ is a restriction of the generator. Let B denote the generator of $\{S(t) : t \geq 0\}$ and $u_0 \in D(A)$. Since the corresponding solution $u(t) = S(t)u_0$ is right-differentiable at 0, we have

$$S(h)u_0 - u_0 = \int_0^h u'(t)\,dt = -\int_0^h Au(t)\,dt \ , \qquad h > 0 \ .$$

Hence, we have $D^+(S(0)u_0) = -Au_0$, so $u_0 \in D(B)$ and $Bu_0 = -Au_0$.

We shall see later that if $-A$ is the generator of a contraction semigroup, then A is accretive, $D(A)$ is dense, and for every $u_0 \in D(A)$ there is a unique solution $u \in C^1([0,\infty), H)$ of (2.1) with $u(0) = u_0$. But first, we consider a simple example.

Theorem 2.2 *For each $B \in \mathcal{L}(H)$, the series $\sum_{n=0}^{\infty}(B^n/n!)$ converges in $\mathcal{L}(H)$; denote its sum by $\exp(B)$. The function $t \mapsto \exp(tB) : \mathbb{R} \to \mathcal{L}(H)$ is infinitely differentiable and satisfies*

$$D[\exp(tB)] = B \cdot \exp(tB) = \exp(tB) \cdot B \ , \qquad t \in \mathbb{R} \ . \tag{2.4}$$

If $B_1, B_2 \in \mathcal{L}(H)$ and if $B_1 \cdot B_2 = B_2 \cdot B_1$, then

$$\exp(B_1 + B_2) = \exp(B_1) \cdot \exp(B_2) \ . \tag{2.5}$$

Proof: The convergence of the series in $\mathcal{L}(H)$ follows from that of $\sum_{n=0}^{\infty} \|B\|_{\mathcal{L}(H)}^n/n! = \exp(\|B\|)$ in $\mathbb{R}$. To verify the differentiability of $\exp(tB)$ at $t = 0$, we note that

$$\Big[(\exp(tB) - I)/t\Big] - B = (1/t)\sum_{n=2}^{\infty}(tB)^n/n! \ , \qquad t \neq 0 \ ,$$

and this gives the estimate

$$\|\Big[(\exp(tB) - I)/t\Big] - B\| \leq (1/|t|)\Big[\exp(|t| \cdot \|B\|) - 1 - |t|\,\|B\|\Big] \ .$$

Since $t \mapsto \exp(t\|B\|)$ is (right) differentiable at 0 with (right) derivative $\|B\|$, it follows that (2.4) holds at $t = 0$. The semigroup property shows that (2.4) holds at every $t \in \mathbb{R}$. (We leave (2.5) as an exercise.)

3 Generation of Semigroups

Our objective here is to characterize those operators which generate contraction semigroups.

To first obtain necessary conditions, we assume that $B : D(B) \to H$ is the generator of a contraction semigroup $\{S(t) : t \geq 0\}$. If $t \geq 0$ and $x \in D(B)$, then the last term in the identity

$$h^{-1}(S(t+h)x - S(t)x) = h^{-1}(S(h) - I)S(t)x = h^{-1}S(t)(S(h)x - x) \ , \qquad h > 0 \ ,$$

has a limit as $h \to 0^+$, hence, so also does each term and we obtain

$$D^+S(t)x = BS(t)x = S(t)Bx \ , \qquad x \in D(B) \ , \ t \geq 0 \ .$$

Similarly, using the uniform boundedness of the semigroup we may take the limit as $h \to 0^+$ in the identity

$$h^{-1}(S(t)x - S(t-h)x) = S(t-h)h^{-1}(S(h)x - x) \ , \qquad 0 < h < t \ ,$$

to obtain

$$D^- S(t)x = S(t)Bx \ , \qquad x \in D(B) \ , \ t > 0 \ .$$

We summarize the above.

Lemma *For each $x \in D(B)$, $S(\cdot)x \in C^1(\mathbb{R}_0^+, H)$, $S(t)x \in D(B)$, and*

$$S(t)x - x = \int_0^t BS(s)x\,ds = \int_0^t S(s)Bx\,dx \ , \qquad t \geq 0 \ . \tag{3.1}$$

Corollary *B is closed.*

Proof: Let $x_n \in D(B)$ with $x_n \to x$ and $Bx_n \to y$ in H. For each $h > 0$ we have from (3.1)

$$h^{-1}(S(h)x_n - x_n) = h^{-1} \int_0^h S(s)Bx_n\,ds \ , \qquad n \geq 1 \ .$$

Letting $n \to \infty$ and then $h \to 0^+$ gives $D^+S(0)x = y$, hence, $Bx = y$.

Lemma *$D(B)$ is dense in H; for each $t \geq 0$ and $x \in H$, $\int_0^t S(s)x\,ds \in D(B)$ and*

$$S(t)x - x = B\int_0^t S(s)x\,ds \ , \qquad x \in H \ , \ t \geq 0 \ . \tag{3.2}$$

Proof: Define $x_t = \int_0^t S(s)x\,ds$. Then for $h > 0$

$$\begin{aligned} h^{-1}(S(h)x_t - x_t) &= h^{-1}\left\{\int_0^t S(h+s)x\,ds - \int_0^t S(s)x\,ds\right\} \\ &= h^{-1}\left\{\int_h^{t+h} S(s)x\,ds - \int_0^t S(s)x\,ds\right\} \ . \end{aligned}$$

Adding and subtracting $\int_t^h S(s)x\,ds$ gives the equation

$$h^{-1}(S(h)x_t - x_t) = h^{-1}\int_t^{t+h} S(s)x\,ds - h^{-1}\int_0^h S(s)x\,ds \ ,$$

and letting $h \to 0$ shows that $x_t \in D(B)$ and $Bx_t = S(t)x - x$. Finally, from $t^{-1}x_t \to x$ as $t \to 0^+$, it follows that $D(B)$ is dense in H.

Let $\lambda > 0$. Then it is easy to check that $\{e^{-\lambda t}S(t) : t \geq 0\}$ is a contraction semigroup whose generator is $B - \lambda$ with domain $D(B)$. From (3.1) and (3.2) applied to this semigroup we obtain

$$e^{-\lambda t}S(t)x - x = \int_0^t e^{-\lambda s}S(s)(B - \lambda)x\,ds\ , \qquad x \in D(B)\ ,\ t \geq 0\ ,$$

$$e^{-\lambda t}S(t)y - y = (B - \lambda)\int_0^t e^{-\lambda s}S(s)y\,ds\ , \qquad y \in H\ , \qquad t \geq 0\ .$$

Letting $t \to \infty$ (and using the fact that B is closed to evaluate the limit of the last term) we find that

$$x = \int_0^\infty e^{-\lambda s}S(s)(\lambda - B)x\,ds\ , \qquad x \in D(B)\ ,$$

$$y = (\lambda - B)\int_0^\infty e^{-\lambda s}S(s)y\,ds\ , \qquad y \in H\ .$$

These identities show that $\lambda - B$ is injective and surjective, respectively, with

$$\|(\lambda - B)^{-1}y\| \leq \int_0^\infty e^{-\lambda s}\,ds\|y\| = \lambda^{-1}\|y\|\ , \qquad y \in H\ .$$

This proves the necessity part of the following fundamental result.

Theorem 3.1 *Necessary and sufficient conditions that the operator $B : D(B) \to H$ be the generator of a contraction semigroup on H are that*

> *$D(B)$ is dense in H and $\lambda - B : D(B) \to H$ is a bijection with $\|\lambda(\lambda - B)^{-1}\|_{\mathcal{L}(H)} \leq 1$ for all $\lambda > 0$.*

Proof: (Continued) It remains to show that the indicated conditions on B imply that it is the generator of a semigroup. We shall achieve this as follows: (a) approximate B by bounded operators, B_λ, (b) obtain corresponding semigroups $\{S_\lambda(t) : t \geq 0\}$ by exponentiating B_λ, then (c) show that $S(t) \equiv \lim_{\lambda\to\infty} S_\lambda(t)$ exists and is the desired semigroup.

Since $\lambda - B : D(B) \to H$ is a bijection for each $\lambda > 0$, we may define $B_\lambda = \lambda B(\lambda - B)^{-1}$, $\lambda > 0$.

Lemma *For each $\lambda > 0$, $B_\lambda \in \mathcal{L}(H)$ and satisfies*

$$B_\lambda = -\lambda + \lambda^2(\lambda - B)^{-1}\ . \tag{3.3}$$

For $x \in D(B)$, $\|B_\lambda(x)\| \le \|Bx\|$ *and* $\lim_{\lambda\to\infty} B_\lambda(x) = Bx$.

Proof: Equation (3.3) follows from $(B_\lambda+\lambda)(\lambda-B)x = \lambda^2 x$, $x \in D(B)$. The estimate is obtained from $B_\lambda = \lambda(\lambda - B)^{-1}B$ and the fact that $\lambda(\lambda - B)^{-1}$ is a contraction. Finally, we have from (3.3)

$$\|\lambda(\lambda - B)^{-1}x - x\| = \|\lambda^{-1}B_\lambda x\| \le \lambda^{-1}\|Bx\| \ , \qquad \lambda > 0 \ , \ x \in D(B) \ ,$$

hence, $\lambda(\lambda-B)^{-1}x \mapsto x$ for all $x \in D(B)$. But $D(B)$ dense and $\{\lambda(\lambda-B)^{-1}\}$ uniformly bounded imply $\lambda(\lambda - B)^{-1}x \to x$ for all $x \in H$, and this shows $B_\lambda x = \lambda(\lambda - B)^{-1}Bx \to Bx$ for $x \in D(B)$.

Since B_λ is bounded for each $\lambda > 0$, we may define by Theorem 2.2

$$S_\lambda(t) = \exp(tB_\lambda) \ , \qquad \lambda > 0 \ , \ t \ge 0 \ .$$

Lemma *For each* $\lambda > 0$, $\{S_\lambda(t) : t \ge 0\}$ *is a contraction semigroup on* H *with generator* B_λ. *For each* $x \in D(B)$, $\{S_\lambda(t)x\}$ *converges in* H *as* $\lambda \to \infty$, *and the convergence is uniform for* $t \in [0, T]$, $T > 0$.

Proof: The first statement follows from

$$\|S_\lambda(t)\| = e^{-\lambda t}\|\exp(\lambda^2(\lambda - B)^{-1}t)\| \le e^{-\lambda t}e^{\lambda t} = 1 \ ,$$

and $D(S_\lambda(t)) = B_\lambda S_\lambda(t)$. Furthermore,

$$\begin{aligned} S_\lambda(t) - S_\mu(t) &= \int_0^t D_s S_\mu(t-s)S_\lambda(s)\,ds \\ &= \int_0^t S_\mu(t-s)S_\lambda(s)(B_\lambda - B_\mu)\,ds \ , \qquad \mu, \lambda > 0 \ , \end{aligned}$$

in $\mathcal{L}(H)$, so we obtain

$$\|S_\lambda(t)x - S_\mu(t)s\| \le t\|B_\lambda x - B_\mu x\| \ , \qquad \lambda, \mu > 0 \ , \ t \ge 0 \ , \ x \in D(B) \ .$$

This shows $\{S_\lambda(t)x\}$ is uniformly Cauchy for t on bounded intervals, so the Lemma follows.

Since each $S_\lambda(t)$ is a contraction and $D(B)$ is dense, the indicated limit holds for all $x \in H$, and uniformly on bounded intervals. We define $S(t)x = \lim_{\lambda\to\infty} S_\lambda(t)x$, $x \in H$, $t \ge 0$, and it is clear that each $S(t)$ is a linear contraction. The uniform convergence on bounded intervals implies $t \mapsto$

$S(t)x$ is continuous for each $x \in H$ and the semigroup identity is easily verified. Thus $\{S(t) : t \geq 0\}$ is a contraction semigroup on H. If $x \in D(B)$ the functions $S_\lambda(\cdot)B_\lambda x$ converge uniformly to $S(\cdot)Bx$ and, hence, for $h > 0$ we may take the limit in the identity

$$S_\lambda(h)x - x = \int_0^h S_\lambda(t)B_\lambda x\, dt$$

to obtain

$$S(h)x - x = \int_0^h S(t)Bx\, dt\ , \qquad x \in D(B)\ ,\ h > 0\ .$$

This implies that $D^+(S(0)x) = Bx$ for $x \in D(B)$. If C denotes the generator of $\{S(t) : t \geq 0\}$, we have shown that $D(B) \subset D(C)$ and $Bx = Cx$ for all $x \in D(B)$. That is, C is an extension of B. But $I - B$ is surjective and $I - C$ is injective, so it follows that $D(B) = D(C)$.

Corollary 3.2 *If $-A$ is the generator of a contraction semigroup, then for each $u_0 \in D(A)$ there is a unique solution $u \in C^1([0,\infty), H)$ of* (2.1) *with $u(0) = u_0$.*

Proof: This follows immediately from (3.1).

Theorem 3.3 *If $-A$ is the generator of a contraction semigroup, then for each $u_0 \in D(A)$ and each $f \in C^1([0,\infty), H)$ there is a unique $u \in C^1([0,\infty), H)$ such that $u(0) = u_0$, $u(t) \in D(A)$ for $t \geq 0$, and*

$$u'(t) + Au(t) = f(t)\ , \qquad t \geq 0\ . \tag{3.4}$$

Proof: It suffices to show that the function

$$g(t) = \int_0^t S(t-\tau)f(\tau)\, d\tau\ , \qquad t \geq 0\ ,$$

satisfies (3.4) and to note that $g(0) = 0$. Letting $z = t - \tau$ we have

$$(g(t+h) - g(t))/h = \int_0^t S(z)(f(t+h-z) - f(t-z))h^{-1}\, dz$$
$$+h^{-1} \int_t^{t+h} S(z)f(t+h-z)\, dz$$

so it follows that $g'(t)$ exists and

$$g'(t) = \int_0^t S(z)f'(t-z)\,dz + S(t)f(0)\ .$$

Furthermore we have

$$\begin{aligned}(g(t+h)-g(t))/h &= h^{-1}\Big\{\int_0^{t+h} S(t+h-\tau)f(\tau)\,d\tau - \int_0^t S(t-\tau)f(\tau)\,d\tau\Big\}\\ &= (S(h)-I)h^{-1}\int_0^t S(t-\tau)f(\tau)\,d\tau\\ &\qquad + h^{-1}\int_t^{t+h} S(t+h-\tau)f(\tau)\,d\tau\ . \end{aligned} \tag{3.5}$$

Since $g'(t)$ exists and since the last term in (3.5) has a limit as $h \to 0^+$, it follows from (3.5) that

$$\int_0^t S(t-\tau)f(\tau)\,d\tau \in D(A)$$

and that g satisfies (3.4).

4 Accretive Operators; two examples

We shall characterize the generators of contraction semigroups among the negatives of accretive operators. In our applications to boundary value problems, the conditions of this characterization will be more easily verified than those of Theorem 3.1. These applications will be illustrated by two examples; the first contains a first order partial differential equation and the second is the second order equation of heat conduction in one dimension. Much more general examples of the latter type will be given in Section 7.

The two following results are elementary and will be used below and later.

Lemma 4.1 *Let $B \in \mathcal{L}(H)$ with $\|B\| < 1$. Then $(I-B)^{-1} \in \mathcal{L}(H)$ and is given by the power series $\sum_{n=0}^{\infty} B^n$ in $\mathcal{L}(H)$.*

Lemma 4.2 *Let $A \in L(D(A),H)$ where $D(A) \le H$, and assume $(\mu - A)^{-1} \in \mathcal{L}(H)$, with $\mu \in \mathbb{C}$. Then $(\lambda - A)^{-1} \in \mathcal{L}(H)$ for $\lambda \in \mathbb{C}$, if and only if $[I-(\mu-\lambda)(\mu-A)^{-1}]^{-1} \in \mathcal{L}(H)$, and in that case we have*

$$(\lambda - A)^{-1} = (\mu - A)^{-1}[I-(\mu-\lambda)(\mu-A)^{-1}]^{-1}\ .$$

Proof: Let $B \equiv I - (\mu - \lambda)(\mu - A)^{-1}$ and assume $B^{-1} \in \mathcal{L}(H)$. Then we have

$$\begin{aligned}(\lambda - A)(\mu - A)^{-1}B^{-1} &= [(\lambda - \mu) + (\mu - A)](\mu - A)^{-1}B^{-1} \\ &= [(\lambda - \mu)(\mu - A)^{-1} + I]B^{-1} = I \ ,\end{aligned}$$

and

$$\begin{aligned}(\mu - A)^{-1}B^{-1}(\lambda - A) &= (\mu - A)^{-1}B^{-1}[(\lambda - \mu) + (\mu - A)] \\ &= (\mu - A)^{-1}B^{-1}[B(\mu - A)] = I \ , \quad \text{on} \quad D(A) \ .\end{aligned}$$

The converse is proved similarly.

Suppose now that $-A$ generates a contraction semigroup on H. From Theorem 3.1 it follows that

$$\|(\lambda + A)x\| \geq \lambda\|x\| \ , \qquad \lambda > 0 \ , \ x \in D(A) \ , \tag{4.1}$$

and this is equivalent to

$$2\,\mathrm{Re}(Ax, x)_H \geq -\|Ax\|^2/\lambda \ , \qquad \lambda > 0 \ , \ x \in D(A) \ .$$

But this shows A is accretive and, hence, that Theorem 3.1 implies the necessity part of the following.

Theorem 4.3 *The linear operator $-A : D(A) \to H$ is the generator of a contraction semigroup on H if and only if $D(A)$ is dense in H, A is accretive, and $\lambda + A$ is surjective* for some $\lambda > 0$.

Proof: (Continued) It remains to verify that the above conditions on the operator A imply that $-A$ satisfies the conditions of Theorem 3.1. Since A is accretive, the estimate (4.1) follows, and it remains to show that $\lambda + A$ is surjective for every $\lambda > 0$.

We are given $(\mu + A)^{-1} \in \mathcal{L}(H)$ for some $\mu > 0$ and $\|\mu(\mu + A)^{-1}\| \leq 1$. For any $\lambda \in C$ we have $\|(\lambda - \mu)(\mu + A)^{-1}\| \leq |\lambda - \mu|/\mu$, hence Lemma 4.1 shows that $I - (\lambda - \mu)(\lambda + A)^{-1}$ has an inverse which belongs to $\mathcal{L}(H)$ if $|\lambda - \mu| < \mu$. But then Lemma 4.2 implies that $(\lambda + A)^{-1} \in \mathcal{L}(H)$. Thus, $(\mu + A)^{-1} \in \mathcal{L}(H)$ with $\mu > 0$ implies that $(\lambda + A)^{-1} \in \mathcal{L}(H)$ for all $\lambda > 0$ such that $|\lambda - \mu| < \mu$, i.e., $0 < \lambda < 2\mu$. The result then follows by induction.

Example 1. Let $H = L^2(0,1)$, $c \in \mathbb{C}$, $D(A) = \{u \in H^1(0,1) : u(0) = cu(1)\}$, and $A = \partial$. Then we have for $u \in H^1(0,1)$

$$2\,\mathrm{Re}(Au, u)_H = \int_0^1 (\partial u \cdot \bar{u} + \overline{\partial u} \cdot u) = |u(1)|^2 - |u(0)|^2 \ .$$

Thus, A is accretive if (and only if) $|c| \leq 1$, and we assume this hereafter. Theorem 4.3 implies that $-A$ generates a contraction semigroup on $L^2(0,1)$ if (and only if) $I + A$ is surjective. But this follows from the solvability of the problem

$$u + \partial u = f \ , \qquad u(0) = cu(1)$$

for each $f \in L^2(0,1)$; the solution is given by

$$u(x) = \int_0^1 G(x,s) f(s)\, ds \ ,$$

$$G(x,s) = \begin{cases} [e/(e-c)]e^{-(x-s)} \ , & 0 \leq s < x \leq 1 \ , \\ [c/(e-c)]e^{-(x-s)} \ , & 0 \leq x < s \leq 1 \ . \end{cases}$$

Since $-A$ generates a contraction semigroup, the initial boundary value problem

$$\partial_t u(x,t) + \partial_x u(x,t) = 0 \ , \qquad 0 < x < 1 \ , \ t \geq 0 \tag{4.2}$$

$$u(0,t) = cu(1,t) \tag{4.3}$$

$$u(x,0) = u_0(x) \tag{4.4}$$

has a unique solution for each $u_0 \in D(A)$. This can be verified directly. Since any solution of (4.2) is locally of the form

$$u(x,t) = F(x-t)$$

for some function F; the equation (4.4) shows

$$u(x,t) = u_0(x-t) \ , \qquad 0 \leq t \leq x \leq 1 \ .$$

Then (4.3) gives $u(0,t) = cu_0(1-t)$, $0 \leq t \leq 1$, so (4.2) then implies

$$u(x,t) = cu_0(1+x-t) \ , \qquad x \leq t \leq x+1 \ .$$

An easy induction gives the representation

$$u(x,t) = c^n u_0(n+x-t) \ , \qquad n-1+x \leq t \leq n+1 \ , \ n \geq 1 \ .$$

The representation of the solution of (4.2)–(4.4) gives some additional information on the solution. First, the Cauchy problem can be solved only if $u_0 \in D(A)$, because $u(\cdot,t) \in D(A)$ implies $u(\cdot,t)$ is (absolutely) continuous and this is possible only if u_0 satisfies the boundary condition (4.3). Second, the solution satisfies $u(\cdot,t) \in H^1(0,1)$ for every $t \geq 1$ but will *not* belong to $H^2(0,1)$ unless $\partial u_0 \in D(A)$. That is, we do not in general have $u(\cdot,t) \in H^2(0,1)$, no matter how smooth the initial function u_0 may be. Finally, the representation above defines a solution of (4.2)–(4.4) on $-\infty < t < \infty$ by allowing n to be any integer. Thus, the problem can be solved backwards in time as well as forward. This is related to the fact that $-A$ generates a *group* of operators and we shall develop this notion in Section 5. Also see Section V.3 and Chapter VI.

Example 2. For our second example, we take $H = L^2(0,1)$ and let $A = -\partial^2$ on $D(A) = H_0^1(0,1) \cap H^2(0,1)$. An integration-by-parts gives

$$(Au,u)_H = \int_0^1 |\partial u|^2 \ , \qquad u \in D(A) \ ,$$

so A is accretive, and the solvability of the boundary value problem

$$u - \partial^2 u = f \ , \qquad u(0) = 0 \ , \ u(1) = 0 \ , \tag{4.5}$$

for $f \in L^2(0,1)$ shows that $I + A$ is surjective. (We may either solve (4.5) directly by the classical variation-of-parameters method, thereby obtaining the representation

$$u(x) = \int_0^1 G(x,s) f(s)\, ds \ ,$$

$$G(x,s) = \begin{cases} \dfrac{\sinh(1-x)\sinh(s)}{\sinh(1)} \ , & 0 \leq s < x \leq 1 \\ \dfrac{\sinh(1-s)\sinh(x)}{\sinh(1)} \ , & 0 \leq x < s \leq 1 \end{cases}$$

or observe that it is a special case of the boundary value problem of Chapter III.) Since $-A$ generates a contraction semigroup on $L^2(0,1)$, it follows from Corollary 3.2 that there is a unique solution of the initial-boundary value problem

$$\begin{aligned} &\partial_t u - \partial_x^2 u = 0 \ , \qquad 0 < x < 1 \ , \ t \geq 0 \\ &u(0,t) = 0 \ , \quad u(1,t) = 0 \ , \\ &u(x,0) = u_0(x) \end{aligned} \tag{4.6}$$

for each initial function $u_0 \in D(A)$.

A representation of the solution of (4.6) can be obtained by the method of separation-of-variables. This representation is the Fourier series (cf. (1.4))

$$u(x,t) = 2\int_0^1 \sum_{n=0}^{\infty} u_0(s)\sin(ns)\sin(nx)e^{-n^2 t}\,ds \tag{4.7}$$

and it gives information that is not available from Corollary 3.2. First, (4.7) defines a solution of the Cauchy problem for every $u_0 \in L^2(0,1)$, not just for those in $D(A)$. Because of the factor $e^{-n^2 t}$ in the series (4.7), every derivative of the sequence of partial sums is convergent in $L^2(0,1)$ whenever $t > 0$, and one can thereby show that the solution is infinitely differentiable in the open cylinder $(0,1)\times(0,\infty)$. Finally, the series will in general not converge if $t < 0$. This occurs because of the exponential terms, and severe conditions must be placed on the initial data u_0 in order to obtain convergence at a point where $t < 0$. Even when a solution exists on an interval $[-T,0]$ for some $T > 0$, it will not depend continuously on the initial data (cf., Exercise 1.3). The preceding situation is typical of Cauchy problems which are resolved by *analytic semigroups*. Such Cauchy problems are (appropriately) called *parabolic* and we shall discuss these notions in Sections 6 and 7 and again in Chapters V and VI.

5 Generation of Groups; a wave equation

We are concerned here with a situation in which the evolution equation can be solved on the whole real line $\mathbb{R}$, not just on the half-line $\mathbb{R}^+$. This is the case when $-A$ generates a *group* of operators on H.

Definition. A *unitary group* on H is a set $\{G(t) : t \in \mathbb{R}\}$ of linear operators on H which satisfy

$$G(t+\tau) = G(t)\cdot G(\tau)\ ,\quad G(0) = I\ ,\qquad t,\tau \in \mathbb{R}\ , \tag{5.1}$$

$$G(\cdot)x \in C(\mathbb{R},H)\ ,\qquad x \in H\ , \tag{5.2}$$

$$\|G(t)\|_{\mathcal{L}(H)} = 1\ ,\qquad t \in \mathbb{R}\ . \tag{5.3}$$

The *generator* of this unitary group is the operator B with domain

$$D(B) = \Big\{x \in H : \lim_{h\to 0} h^{-1}(G(h)-I)x \ \text{ exists in } \ H\Big\}$$

with values given by $Bx = \lim_{h\to 0} h^{-1}(G(h) - I)x = D(G(0)x)$, the (two-sided) derivative at 0 of $G(t)x$.

Equation (5.1) is the group condition, (5.2) is the condition of strong continuity of the group, and (5.3) shows that each operator $G(t)$, $t \in \mathbb{R}$, is an isometry. Note that (5.1) implies

$$G(t) \cdot G(-t) = I \ , \qquad t \in \mathbb{R} \ ,$$

so each $G(t)$ is a bijection of H onto H whose inverse is given by

$$G^{-1}(t) = G(-t) \ , \qquad t \in \mathbb{R} \ .$$

If $B \in \mathcal{L}(H)$, then (5.1) and (5.2) are satisfied by $G(t) \equiv \exp(tB)$, $t \in \mathbb{R}$ (cf., Theorem 2.2). Also, it follows from (2.4) that B is the generator of $\{G(t) : t \in \mathbb{R}\}$ and

$$D(\|G(t)x\|^2) = 2\,\mathrm{Re}(BG(t)x, G(t)x)_H \ , \qquad x \in H \ , \ t \in \mathbb{R} \ ,$$

hence, (5.3) is satisfied if and only if $\mathrm{Re}(Bx, x)_H = 0$ for all $x \in H$. These remarks lead to the following.

Theorem 5.1 *The linear operator $B : D(B) \to H$ is the generator of a unitary group on H if and only if $D(B)$ is dense in H and $\lambda - B$ is a bijection with $\|\lambda(\lambda - B)^{-1}\|_{\mathcal{L}(H)} \leq 1$ for all $\lambda \in \mathbb{R}$, $\lambda \neq 0$.*

Proof: If B is the generator of the unitary group $\{G(t) : t \in \mathbb{R}\}$, then B is the generator of the contraction semigroup $\{G(t) : t \geq 0\}$ and $-B$ is the generator of the contraction semigroup $\{G(-t) : t \geq 0\}$. Thus, both B and $-B$ satisfy the necessary conditions of Theorem 3.1, and this implies the stated conditions on B. Conversely, if B generates the contraction semigroup $\{S_+(t) : t \geq 0\}$ and $-B$ generates the contraction semigroup $\{S_-(t) : t \geq 0\}$, then these operators commute. For each $x_0 \in D(B)$ we have

$$D[S_+(t)S_-(-t)x_0] = 0 \ , \qquad t \geq 0 \ ,$$

so $S_+(t)S_-(-t) = I$, $t \geq 0$. This shows that the family of operators defined by

$$G(t) = \begin{cases} S_+(t) \ , & t \geq 0 \\ S_-(-t) \ , & t < 0 \end{cases}$$

satisfies (5.1). The condition (5.2) is easy to check and (5.3) follows from

$$1 = \|G(t) \cdot G(-t)\| \leq \|G(t)\| \cdot \|G(-t)\| \leq \|G(t)\| \leq 1 \ .$$

Finally, it suffices to check that B is the generator of $\{G(t) : t \in \mathbb{R}\}$ and then the result follows.

Corollary 5.2 *The operator A is the generator of a unitary group on H if and only if for each $u_0 \in D(A)$ there is a unique solution $u \in C^1(\mathbb{R}, H)$ of (2.1) with $u(0) = u_0$ and $\|u(t)\| = \|u_0\|$, $t \in \mathbb{R}$.*

Proof: This is immediate from the proof of Theorem 5.1 and the results of Theorem 2.1 and Corollary 3.2.

Corollary 5.3 *If A generates a unitary group on H, then for each $u_0 \in D(A)$ and each $f \in C^1(\mathbb{R}, H)$ there is a unique solution $u \in C^1(\mathbb{R}, H)$ of (3.3) and $u(0) = u_0$. This solution is given by*

$$u(t) = G(t)u_0 + \int_0^t G(t-\tau)f(\tau)\,d\tau \ , \qquad t \in \mathbb{R} \ .$$

Finally, we obtain an analogue of Theorem 4.3 by noting that both $+A$ and $-A$ are accretive exactly when A satisfies the following.

Definition. The linear operator $A \in L(D(A), H)$ is said to be *conservative* if

$$\mathrm{Re}(Ax, x)_H = 0 \ , \qquad x \in D(A) \ .$$

Corollary 5.4 *The linear operator $A : D(A) \to H$ is the generator of a unitary group on H if and only if $D(A)$ is dense in H, A is conservative, and $\lambda + A$ is surjective for some $\lambda > 0$ and for some $\lambda < 0$.*

Example. Take $H = L^2(0,1) \times L^2(0,1)$, $D(A) = H_0^1(0,1) \times H^1(0,1)$, and define

$$A[u,v] = [-i\partial v, i\partial u] \ , \qquad [u,v] \in D(A) \ .$$

Then we have

$$(A[u,v],[u,v])_H = i\int_0^1 (\partial v \cdot \bar{u} - \partial u \cdot \bar{v}) \ , \qquad [u,v] \in D(A)$$

and an integration-by-parts gives

$$2\,\mathrm{Re}(A[u,v],[u,v])_H = i(\,\bar{u}(x)v(x) - u(x)\bar{v}(x))\Big|_{x=0}^{x=1} = 0\ , \tag{5.4}$$

since $u(0) = u(1) = 0$. Thus, A is a conservative operator. If $\lambda \neq 0$ and $[f_1, f_2] \in H$, then

$$\lambda[u,v] + A[u,v] = [f_1, f_2]\ , \qquad [u,v] \in D(A)$$

is equivalent to the system

$$-\partial^2 u + \lambda^2 u = \lambda f_1 - i\partial f_2\ , \qquad u \in H_0^1(0,1)\ , \tag{5.5}$$

$$-i\partial u + \lambda v = f_2\ , \qquad v \in H^1(0,1)\ . \tag{5.6}$$

But (5.5) has a unique solution $u \in H_0^1(0,1)$ by Theorem III.2.2 since $\lambda f_1 - i\partial f_2 \in (H_0^1)'$ from Theorem II.2.2. Then (5.6) has a solution $v \in L^2(0,1)$ and it follows from (5.6) that

$$(i\lambda)\partial v = \lambda f_1 - \lambda^2 u \in L^2(0,1)\ ,$$

so $v \in H^1(0,1)$. Thus $\lambda + A$ is surjective for $\lambda \neq 0$.

Corollaries 5.3 and 5.4 imply that the Cauchy problem

$$\begin{aligned} D\mathbf{u}(t) + A\mathbf{u}(t) &= [0, f(t)]\ , \qquad t \in \mathbb{R}\ , \\ \mathbf{u}(0) &= [u_0, v_0] \end{aligned} \tag{5.7}$$

is well-posed for $u_0 \in H_0^1(0,1)$, $v_0 \in H^1(0,1)$, and $f \in C^1(\mathbb{R}, H)$. Denoting by $u(t)$, $v(t)$, the components of $\mathbf{u}(t)$, i.e., $\mathbf{u}(t) \equiv [u(t), v(t)]$, it follows that $u \in C^2(\mathbb{R}, L^2(0,1))$ satisfies the wave equation

$$\partial_t^2 u(x,t) - \partial_x^2 u(x,t) = f(x,t)\ , \qquad 0 < x < 1\ ,\ t \in \mathbb{R}\ ,$$

and the initial-boundary conditions

$$\begin{aligned} u(0,t) &= u(1,t) = 0 \\ u(x,0) &= u_0(x)\ , \quad \partial_t u(x,0) = -iv_0(x)\ . \end{aligned}$$

See Section VI.5 for additional examples of this type.

6 Analytic Semigroups

We shall consider the Cauchy problem for the equation (2.1) in the special case in which A is a model of an elliptic boundary value problem (cf. Corollary 3.2). Then (2.1) is a corresponding abstract parabolic equation for which Example 2 of Section IV.4 was typical. We shall first extend the definition of $(\lambda + A)^{-1}$ to a sector properly containing the right half of the complex plane $\mathbb{C}$ and then obtain an integral representation for an analytic continuation of the semigroup generated by $-A$.

Theorem 6.1 *Let V and H be Hilbert spaces for which the identity $V \hookrightarrow H$ is continuous. Let $a : V \times V \to \mathbb{C}$ be continuous, sesquilinear, and V-elliptic. In particular*

$$|a(u,v)| \le K\|u\|\,\|v\| \ , \qquad u, v \in V \ ,$$

$$\operatorname{Re} a(u,u) \ge c\|u\|^2 \ , \qquad u \in V \ ,$$

where $0 < c \le K$. Define

$$D(A) = \{u \in V : |a(u,v)| \le K_u |v|_H \ , \ v \in V\} \ ,$$

where K_u depends on u, and let $A \in L(D(A), H)$ be given by

$$a(u,v) = (Au, v)_H \ , \qquad u \in D(A) \ , \ v \in V \ .$$

Then $D(A)$ is dense in H and there is a θ_0, $0 < \theta_0 < \pi/4$, such that for each $\lambda \in S(\pi/2 + \theta_0) \equiv \{z \in \mathbb{C} : |\arg(z)| < \pi/2 + \theta_0\}$ we have $(\lambda + A)^{-1} \in \mathcal{L}(H)$. For each θ, $0 < \theta < \theta_0$, there is an M_θ such that

$$\|\lambda(\lambda + A)^{-1}\|_{\mathcal{L}(H)} \le M_\theta \ , \qquad \lambda \in S(\theta + \pi/2) \ . \tag{6.1}$$

Proof: Suppose $\lambda \in \mathbb{C}$ with $\lambda = \sigma + i\tau$, $\sigma \ge 0$. Since the form u, $v \mapsto a(u,v) + \lambda(u,v)_H$ is V-elliptic it follows that $\lambda + A : D(A) \to H$ is surjective. (This follows directly from the discussion in Section III.2.2; note that A is the restriction of $\mathcal{A}$ to $D(A) = D$.) Furthermore we have the estimate

$$\sigma |u|_H^2 \le \operatorname{Re}\{a(u,u) + \lambda(u,u)_H\} \le |(A + \lambda)u|_H |u|_H \ , \qquad u \in D(A) \ ,$$

so it follows that

$$\|\sigma(\lambda + A)^{-1}\| \le 1 \ , \qquad \operatorname{Re}(\lambda) = \sigma \ge 0 \ . \tag{6.2}$$

From the triangle inequality we obtain

$$|\tau|\,|u|_H^2 - K\|u\|_V^2 \le |\operatorname{Im}((\lambda + A)u, u)_H| \ , \qquad u \in D(A) \ , \tag{6.3}$$

where $\tau = \operatorname{Im}(\lambda)$. We show that this implies that either

$$|\operatorname{Im}((\lambda + A)u, u)_H| \ge (|\tau|/2)|u|_H^2 \tag{6.4}$$

or that

$$\operatorname{Re}((\lambda + A)u, u)_H \ge (c/2K)|\tau|\,|u|_H^2 \ . \tag{6.5}$$

If (6.4) does not hold, then substitution of its negation into (6.3) gives $(|\tau|/2)\|u\|_H^2 \le K\|u\|_V^2$. But we have

$$\operatorname{Re}((\lambda + A)u, u)_H \ge c\|u\|_V^2$$

so (6.5) follows. Since one of (6.4) or (6.5) holds, it follows that

$$|((\lambda + A)u, u)_H| \ge (c/2K)|\tau|\,|u|_H^2 \ , \qquad u \in D(A) \ ,$$

and this gives the estimate

$$\|\tau(\lambda + A)^{-1}\| \le 2K/c \ , \qquad \lambda = \sigma + i\tau \ , \ \sigma \ge 0 \ . \tag{6.6}$$

Now let $\lambda = \sigma + i\tau \in \mathbb{C}$ with $\tau \ne 0$ and set $\mu = i\tau$. From (6.6) we have

$$\|(\mu + A)^{-1}\| \le 2K/c|\mu| \ ,$$

so Lemma 4.1 shows that

$$\|[I - (\lambda - \mu)(\mu + A)^{-1}]^{-1}\| \le [1 - |\lambda - \mu| 2K/c|\mu|]^{-1}$$

whenever $|\sigma|/|\tau| = (\lambda - \mu)/|\mu| < c/2K$.

From Lemma 4.2 we then obtain $(\lambda + A)^{-1} \in \mathcal{L}(H)$ and

$$\|\lambda(\lambda + A)^{-1}\| \le (2K/c)(|\sigma|/|\tau| + 1)(1 - 2K|\sigma|/c|\tau|)^{-1} \ ,$$
$$\lambda = \sigma + i\tau \ , \quad |\sigma|/|\tau| < c/2K \ . \tag{6.7}$$

Theorem 6.1 now follows from (6.2) and (6.7) with $\theta_0 = \tan^{-1}(c/2K)$.

From (6.2) it is clear that the operator $-A$ is the generator of a contraction semigroup $\{S(t) : t \ge 0\}$ on H. We shall obtain an analytic extension of this semigroup.

Theorem 6.2 *Let $A \in L(D(A), H)$ be the operator of Theorem 6.1. Then there is a family of operators $\{T(t) : t \in S(\theta_0)\}$ satisfying*

(a) $T(t+\tau) = T(t) \cdot T(\tau)$, $\qquad t, \tau \in S(\theta_0)$,

and for $x, y \in H$, the function $t \mapsto (T(t)x, y)_H$ is analytic on $S(\theta_0)$;

(b) *for $t \in S(\theta_0)$, $T(t) \in L(H, D(A))$ and*

$$-\frac{dT(t)}{dt} = A \cdot T(t) \in \mathcal{L}(H) \ ;$$

(c) *if $0 < \varepsilon < \theta_0$, then for some constant $C(\varepsilon)$,*

$$\|T(t)\| \leq C(\varepsilon)\|tAT(t)\| \leq C(\varepsilon) \ , \qquad t \in S(\theta_0 - \varepsilon) \ ,$$

and for $x \in H$, $T(t) \to x$ as $t \to 0$, $t \in S(\theta_0 - \varepsilon)$.

Proof: Let θ be chosen with $\theta_0/2 < \theta < \theta_0$ and let C be the path consisting of the two rays $|\arg(z)| = \pi/2 + \theta$, $|z| \geq 1$, and the semi-circle $\{e^{it} : |t| \leq \theta + \pi/2\}$ oriented so as to run from $\infty \cdot e^{-i(\pi/2+\theta)}$ to $\infty \cdot e^{i(\pi/2+\theta)}$.

If $t \in S(2\theta - \theta_0)$, then we have

$$|\arg(\lambda t)| \geq |\arg \lambda| - |\arg t| \geq \pi/2 + (\theta_0 - \theta) \ , \qquad \lambda \in C \ , \ |\lambda| \geq 1 \ ,$$

so we obtain the estimate

$$\mathrm{Re}(\lambda t) \leq -\sin(\theta_0 - \theta)|\lambda t| \ , \qquad t \in S(2\theta - \theta_0) \ .$$

This shows that the (improper) integral

$$T(t) \equiv \frac{1}{2\pi i} \int_C e^{\lambda t} (\lambda + A)^{-1} \, d\lambda \ , \qquad t \in S(2\theta - \theta_0) \tag{6.8}$$

exists and is absolutely convergent in $\mathcal{L}(H)$. If $x, y \in H$ then

$$(T(t)x, y)_H = \frac{1}{2\pi i} \int_C e^{\lambda t} ((\lambda + A)^{-1} x, y)_H \, d\lambda$$

is analytic in t. If C' is a curve obtained by translating C to the right, then from Cauchy's theorem we obtain

$$(T(t)x, y)_H = \frac{1}{2\pi i} \int_{C'} e^{\lambda' t} ((\lambda' + A)^{-1} x, y)_H \, d\lambda' \ .$$

Hence, we have

$$T(t) = \frac{1}{2\pi i} \int_{C'} e^{\lambda' t} (\lambda' + A)^{-1} \, d\lambda' \ ,$$

since x, y are arbitrary and the integral is absolutely convergent in $\mathcal{L}(H)$. The semigroup identity follows from the calculation

$$\begin{aligned}
T(t)T(\tau) &= \left(\frac{1}{2\pi i}\right)^2 \int_{C'} \int_C e^{\lambda'\tau + \lambda t} (\lambda' + A)^{-1} (\lambda + A)^{-1} \, d\lambda \, d\lambda' \\
&= \left(\frac{1}{2\pi i}\right)^2 \left[\int_{C'} e^{\lambda'\tau} (\lambda' + A)^{-1} \left\{ \int_C e^{\lambda t} (\lambda - \lambda')^{-1} \, d\lambda \right\} d\lambda' \right. \\
&\qquad \left. - \int_C e^{\lambda t} (\lambda + A)^{-1} \left\{ \int_{C'} e^{\lambda'\tau} (\lambda - \lambda')^{-1} \, d\lambda' \right\} d\lambda \right] \\
&= \frac{1}{2\pi i} \int_C e^{\lambda(t+\tau)} (\lambda + A)^{-1} \, d\lambda = T(t + \tau) \ ,
\end{aligned}$$

where we have used Fubini's theorem and the identities

$$(\lambda + A)^{-1} (\lambda' + A)^{-1} = (\lambda - \lambda')^{-1} [(\lambda' + A)^{-1} - (\lambda + A)^{-1}] \ ,$$

$$\int_C e^{\lambda t} (\lambda - \lambda')^{-1} \, d\lambda = 0 \quad , \quad \int_{C'} e^{\lambda'\tau} (\lambda - \lambda')^{-1} d\lambda' = -2\pi i e^{\lambda\tau} \ .$$

Since $\theta \in (\theta_0/2, \theta_0)$ is arbitrary, (a) follows from above.

Similarly, we may differentiate (6.8) and obtain

$$\begin{aligned}
\frac{dT(t)}{dt} &= \frac{1}{2\pi i} \int_C e^{\lambda t} \lambda (\lambda + A)^{-1} \, d\lambda \qquad (6.9) \\
&= \frac{1}{2\pi i} \int_C e^{\lambda t} [I - A(\lambda + A)^{-1}] \, d\lambda \\
&= \frac{-1}{2\pi i} \int_C e^{\lambda t} A (\lambda + A)^{-1} \, d\lambda \ .
\end{aligned}$$

Since A is closed, this implies that for $t \in S(2\theta - \theta_0)$, $\theta_0/2 < \theta < \theta_0$, we have

$$-\frac{dT(t)}{dt} = AT(t) \in \mathcal{L}(H)$$

so (b) follows.

We next consider (c). Letting $\theta = \theta_0 - \varepsilon/2$, we obtain from (6.1) and (6.8) the estimate

$$\begin{aligned}\|T(t)\| &\leq \frac{1}{2\pi}\int_C |e^{\lambda t}| \cdot \|(\lambda + A)^{-1}\|\, d|\lambda| \\ &\leq \frac{M_\theta}{2\pi}\int_C e^{\operatorname{Re}\lambda t}\frac{d|\lambda|}{|\lambda|}\ .\end{aligned}$$

Since $\operatorname{Re}\lambda t \leq -\sin(\varepsilon/2)\cdot|\lambda t|$ in this integral, the last quantity depends only on ε. The second estimate in (c) follows similarly.

To study the behavior of $T(t)$ for $t \in S(\theta_0 - \varepsilon)$ close to 0, we first note that if $x \in D(A)$

$$\begin{aligned}T(t)x - x &= \frac{1}{2\pi i}\int_C e^{\lambda t}((\lambda + A)^{-1} - \lambda^{-1})x\, d\lambda \\ &= \frac{-1}{2\pi i}\int_C e^{\lambda t}(\lambda + A)^{-1}Ax\, d\lambda/\lambda\ ,\end{aligned}$$

and, hence, we obtain the estimate

$$\|T(t)x - x\| \leq |t|\frac{M_\theta}{2\pi}\left\{\int_C e^{-\sin(\varepsilon/2)|\lambda t|}\,\frac{d|t\lambda|}{|t\lambda|^2}\right\}\|Ax\|\ .$$

Thus, $T(t)x \to x$ as $t \to 0$ with $t \in S(\theta_0 - \varepsilon)$. Since $D(A)$ is dense and $\{T(t) : t \in S(\theta_0 - \varepsilon)\}$ is uniformly bounded, this proves (c).

Definition. A family of operators $\{T(t) : t \in S(\theta_0) \cup \{0\}\}$ which satisfies the properties of Theorem 6.2 and $T(0) = I$ is called an *analytic semigroup*.

Theorem 6.3 *Let A be the operator of Theorem 6.1, $\{T(t) : t \in S(\theta_0)\}$ be given by (6.8), and $T(0) = I$. Then the collection of operators $\{T(t) : t \geq 0\}$ is the contraction semigroup generated by $-A$.*

Proof: Let $u_0 \in H$ and define $u(t) = T(t)u_0$, $t \geq 0$. Theorem 6.2 shows that u is a solution of the Cauchy problem (2.1) with $u(0) = u_0$. Theorem 2.1 implies that $\{T(t) : t \geq 0\}$ is a contraction semigroup whose generator is an extension of $-A$. But $I + A$ is surjective, so the result follows.

Corollary 6.4 *If A is the operator of Theorem 6.1, then for every $u_0 \in H$ there is a unique solution $u \in C([0,\infty), H) \cap C^\infty((0,\infty), H)$ of (2.1) with $u(0) = u_0$. For each $t > 0$, $u(t) \in D(A^p)$ for every $p \geq 1$.*

There are some important differences between Corollary 6.4 and its counterpart, Corollary 3.2. In particular we note that Corollary 6.4 solves the Cauchy problem for all initial data in H, while Corollary 3.2 is appropriate only for initial data in $D(A)$. Also, the infinite differentiability of the solution from Corollary 6.4 and the consequential inclusion in the domain of every power of A at each $t > 0$ are properties not generally true in the situation of Corollary 3.2. These regularity properties are typical of parabolic problems (cf., Section 7).

Theorem 6.5 *If A is the operator of Theorem 6.1, then for each $u_0 \in H$ and each* Hölder continuous $f : [0, \infty) \to H$*:*

$$\|f(t) - f(\tau)\| \leq K(t - \tau)^{\alpha} , \qquad 0 \leq \tau \leq t ,$$

where K and α are constant, $0 < \alpha \leq 1$, there is a unique $u \in C([0, \infty), H) \cap C^1((0, \infty), H)$ such that $u(0) = u_0$, $u(t) \in D(A)$ for $t > 0$, and

$$u'(t) + Au(t) = f(t) , \qquad t > 0 .$$

Proof: It suffices to show that the function

$$g(t) = \int_0^t T(t - \tau)f(\tau)\, d\tau , \qquad t \geq 0 ,$$

is a solution of the above with $u_0 = 0$. Note first that for $t > 0$

$$g(t) = \int_0^t T(t - \tau)(f(\tau) - f(t))\, d\tau + \int_0^t T(t - \tau)\, d\tau \cdot f(t) .$$

from Theorem 6.2(c) and the Hölder continuity of f we have

$$\|A \cdot T(t - \tau)(f(\tau) - f(t))\| \leq C(\theta_0)K|t - \tau|^{\alpha - 1} ,$$

and since A is closed we have $g(t) \in D(A)$ and

$$Ag(t) = A \int_0^t T(t - \tau)(f(\tau) - f(t))\, d\tau + (I - T(t)) \cdot f(t) .$$

The result now follows from the computation (3.5) in the proof of Theorem 3.3.

7 Parabolic Equations

We were led to consider the abstract Cauchy problem in a Hilbert space H

$$u'(t) + Au(t) = f(t)\ , \qquad t > 0\ ;\ u(0) = u_0 \tag{7.1}$$

by an initial-boundary value problem for the parabolic partial differential equation of heat conduction. Some examples of (7.1) will be given in which A is an operator constructed from an abstract boundary value problem. In these examples A will be a linear unbounded operator in the Hilbert space $L^2(G)$ of equivalence classes of functions on the domain G, so the construction of a representative $U(\cdot, t)$ of $u(t)$ is non-trivial. In particular, if such a representative is chosen arbitrarily, the functions $t \mapsto U(x, t)$ need not even be measurable for a given $x \in G$.

We begin by constructing a measurable representative $U(\cdot, \cdot)$ of a solution $u(\cdot)$ of (7.1) and then make precise the correspondence between the vector-valued derivative $u'(t)$ and the partial derivative $\partial_t U(\cdot, t)$.

Theorem 7.1 *Let $I = [a, b]$, a closed interval in $\mathbb{R}$ and G be an open (or measurable) set in $\mathbb{R}^n$.*

(a) *If $u \in C(I, L^2(G))$, then there is a measurable function $U : I \to \mathbb{R}$ such that*

$$u(t) = U(\cdot, t)\ , \qquad t \in I\ . \tag{7.2}$$

(b) *If $u \in C^1(I, L^2(G))$, U and V are measurable real-valued functions on $G \times I$ for which (7.2) holds for a.e. $t \in I$ and*

$$u'(t) = V(\cdot, t)\ , \qquad a.e.\ \ t \in I\ ,$$

then $V = \partial_t U$ in $\mathcal{D}^*(G \times I)$.

Proof: (a) For each $t \in I$, let $U_0(\cdot, t)$ be a representative of $u(t)$. For each integer $n \geq 1$, let $a = t_0 < t_1 < \cdots < t_n = b$ be the uniform partition of I and define

$$U_n(x, t) = \begin{cases} U_0(x, t_k)\ , & t_k \leq t < t_{k+1}\ , k = 0, 1, \ldots, n-1 \\ U_0(x, t)\ , & t = t_n\ . \end{cases}$$

Then $U_n : G \times I \to \mathbb{R}$ is measurable and

$$\lim_{n\to\infty} \|U_n(\cdot, t) - u(t)\|_{L^2(G)} = 0\ ,$$

uniformly for $t \in I$. This implies

$$\lim_{m,n\to\infty} \int_I \int_G |U_m - U_n|^2 \, dx \, dt = 0$$

and the completeness of $L^2(G \times I)$ gives a $U \in L^2(G \times I)$ for which

$$\lim_{n\to\infty} \int_I \int_G |U - U_n|^2 \, dx \, dt = 0 \ .$$

It follows from the above (and the triangle inequality)

$$\int_I \|u(t) - U(\cdot, t)\|^2_{L^2(G)} \, dt = 0$$

so $u(t) = U(\cdot, t)$ for a.e. $t \in I$. The desired result follows by changing $u(t)$ to $U_0(\cdot, t)$ on a set in I of zero measure.

(b) Let $\Phi \in C_0^\infty(G \times I)$. Then $\varphi(t) \equiv \Phi(\cdot, t)$ defines $\varphi \in C_0^\infty(I, L^2(G))$. But for any $\varphi \in C_0^\infty(I, L^2(G))$ and u as given

$$-\int_I (u(t), \varphi'(t))_{L^2(G)} \, dt = \int_I (u'(t), \varphi(t))_{L^2(G)} \, dt \ ,$$

and thus we obtain

$$-\int_I \int_G U(x,t) D_t \Phi(x,t) \, dx \, dt = \int_I \int_G V(x,t) \Phi(x,t) \, dx \, dt \ .$$

This holds for all $\Phi \in C_0^\infty(G \times I)$, so the stated result holds.

We next consider the construction of the operator A appearing in (7.1) from the abstract boundary value problem of Section III.3. Assume we are given Hilbert spaces $V \subset H$, and B with a linear surjection $\gamma : V \to B$ with kernel V_0. Assume γ factors into an isomorphism of V/V_0 onto B, the injection $V \hookrightarrow H$ is continuous, and V_0 is dense in H, and H is identified with H'. (Thus, we obtain the continuous injections $V_0 \hookrightarrow H \hookrightarrow V_0'$ and $V \hookrightarrow H \hookrightarrow V'$.) (Cf. Section III.2.3 for a typical example.)

Suppose we are given a continuous sesquilinear form $a_1 : V \times V \to \mathbb{K}$ and define the formal operator $A_1 \in \mathcal{L}(V, V_0')$ by

$$A_1 u(v) = a_1(u, v) \ , \qquad u \in V \ , \ v \in V_0 \ .$$

Let $D_0 \equiv \{u \in V : A_1(u) \in H\}$ and denote by $\partial_1 \in L(D_0, B')$ the abstract Green's operator constructed in Theorem III.2.3. Thus

$$a_1(u, v) - (A_1 u, v)_H = \partial_1 u(\gamma(v)) \ , \qquad u \in D_0 \ , \ v \in V \ .$$

Suppose we are also given a continuous sesquilinear form $a_2 : B \times B \to \mathbb{K}$ and define $\mathcal{A}_2 \in \mathcal{L}(B, B')$ by

$$\mathcal{A}_2 u(v) = a_2(u, v) \ , \qquad u, v \in B \ .$$

Then we define a continuous sesquilinear form on V by

$$a(u, v) \equiv a_1(u, v) + a_2(\gamma(u), \gamma(v)) \ , \qquad u, v \in V \ .$$

Consider the triple $\{a(\cdot,\cdot), V, H\}$ above. From these we construct as in Section 6 an unbounded operator on H whose domain $D(A)$ is the set of all $u \in V$ such that there is an $F \in H$ for which

$$a(u, v) = (F, v)_H \ , \qquad v \in V \ .$$

Then define $A \in L(D(A), H)$ by $Au = F$. (Thus, A is the operator in Theorem 6.1.) From Corollary III.3.2 we can obtain the following result.

Theorem 7.2 *Let the spaces, forms and operators be as given above. Then $D(A) \subset D_0$, $A = A_1|_{D(A)}$, and $u \in D(A)$ if and only if $u \in V$, $A_1 u \in H$, and $\partial_1 u + \mathcal{A}_2(\gamma(u)) = 0$ in B'.*

(We leave a direct proof as an exercise.) We obtain the existence of a weak solution of a mixed initial-boundary value problem for a large class of parabolic boundary value problems from Theorems 6.5, 7.1 and 7.2.

Theorem 7.3 *Suppose we are given an abstract boundary value problem as above (i.e., Hilbert spaces V, H, B, continuous sesquilinear forms $a_1(\cdot,\cdot)$, $a_2(\cdot,\cdot)$, and operators γ, ∂_1, A_1 and $\mathcal{A}_2$) and that $H = L^2(G)$ where G is an open set in $\mathbb{R}^n$. Assume that for some $c > 0$*

$$\mathrm{Re}\Big\{a_1(v, v) + a_2(\gamma(v), \gamma(v))\Big\} \geq c\|v\|_V^2 \ , \qquad v \in V \ .$$

Let $U_0 \in L^2(G)$ and a measurable $F : G \times [0, T] \to \mathbb{K}$ be given for which $F(\cdot, t) \in L^2(G)$ for all $t \in [0, T]$ and for some $K \in L^2(G)$ and α, $0 < \alpha \leq 1$, we have

$$|F(x, t) - F(x, \tau)| \leq K(x)|t - \tau|^\alpha \ , \quad \text{a.e. } \ x \in G \ , \ t \in [0, T] \ .$$

Then there exists a $U \in L^2(G \times [0, T])$ such that for all $t > 0$

$$\left.\begin{aligned} &U(\cdot, t) \in V \ , \ \partial_t U(\cdot, t) + A_1 U(\cdot, t) = F(\cdot, t) \ \ \text{in} \ \ L^2(G) \ , \\ &\text{and} \ \ \partial_1 U(\cdot, t) + \mathcal{A}_2(\gamma U(\cdot, t)) = 0 \ \ \text{in} \ \ B' \ , \end{aligned}\right\} \qquad (7.3)$$

and

$$\lim_{t\to 0}\int_G |U(x,t)-U_0(x)|^2\,dx = 0\ .$$

We shall give some examples which illustrate particular cases of Theorem 7.3. Each of the following corresponds to an elliptic boundary value problem in Section III.4, and we refer to that section for details on the computations.

7.1

Let the open set G in $\mathbb{R}^n$, coefficients $a_{ij},\ a_j \in L^\infty(G)$, and sesquilinear form $a(\cdot,\cdot) = a_1(\cdot,\cdot)$, and spaces H and B be given as in Section III.4.1. Let $U_0 \in L^2(G)$ be given together with a function $F : G \times [0,T] \to \mathbb{K}$ as in Theorem 7.3. If we choose

$$V = \{v \in H^1(G) : \gamma_0 v(s) = 0\ ,\quad \text{a.e.}\quad s \in \Gamma\}$$

where Γ is a prescribed subset of ∂G, then a solution U of (7.3) satisfies

$$\left.\begin{aligned}
&\partial_t U - \sum_{i,j=1}^n \partial_j(a_{ij}\partial_i U) + \sum_{j=0}^n a_j\partial_j U = F \quad\text{in}\quad L^2(G\times[0,T])\ ,\\
&U(s,t) = 0\ ,\qquad t>0\ ,\quad \text{a.e.}\quad s\in\Gamma\ ,\quad\text{and}\\
&\frac{\partial U(s,t)}{\partial\nu_A} = 0\ ,\qquad t>0\ ,\quad \text{a.e.}\quad s\in\partial G\sim\Gamma\ ,
\end{aligned}\right\}\tag{7.4}$$

where

$$\frac{\partial U}{\partial \nu_A} \equiv \sum_{i=1}^n \partial_i U\Bigl(\sum_{j=1}^n a_{ij}\nu_j\Bigr)$$

denotes the derivative in the direction determined by $\{a_{ij}\}$ and the unit outward normal ν on ∂G. The second equation in (7.4) is called the boundary condition of *first type* and the third equation is known as the boundary condition of *second type*.

7.2

Let V be a closed subspace of $H^1(G)$ to be chosen below, $H = L^2(G)$, $V_0 = H_0^1(G)$ and define

$$a_1(u,v) = \int_G \nabla u\cdot\overline{\nabla v}\ ,\qquad u,v\in V\ .$$

Then $A_1 = -\Delta_n$ and ∂_1 is an extension of the normal derivative $\partial/\partial\nu$ on ∂G. Let $\alpha \in L^\infty(\partial G)$ and define

$$a_2(\varphi,\psi) = \int_{\partial G} \alpha(s)\varphi(s)\overline{\psi(s)}\,ds\ , \qquad \varphi,\psi \in L^2(\partial G)\ .$$

(Note that $B \subset L^2(\partial G) \subset B'$ and $\mathcal{A}_2\varphi = \alpha\cdot\varphi$.) Let $U_0 \in L^2(G)$ and F be given as in Theorem 7.3. Then (exercise) Theorem 7.3 asserts the existence of a solution of (7.3). If we choose $V = H^1(G)$, this solution satisfies

$$\left.\begin{aligned} &\partial_t U - \Delta_n U = F \quad \text{in} \quad L^2(G\times[0,T])\ , \\ &\frac{\partial U(s,t)}{\partial\nu} + \alpha(s)U(s,t) = 0\ , \qquad t>0\ , \quad \text{a.e.} \quad s\in\partial G \end{aligned}\right\} \tag{7.5}$$

If we choose $V = \{v \in H^1(G) : \gamma v = \text{constant}\}$, then U satisfies

$$\left.\begin{aligned} &\partial_t U - \Delta_n U = F \quad \text{in} \quad L^2(G\times[0,T])\ , \\ &U(s,t) = u_0(t)\ , \qquad t>0\ , \quad \text{a.e.} \quad s\in\partial G\ , \\ &\int_{\partial G}\frac{\partial U(s,t)}{\partial\nu}\,ds + \int_{\partial G}\alpha(s)\,ds\cdot u_0(t) = 0\ , \qquad t>0\ . \end{aligned}\right\} \tag{7.6}$$

The boundary conditions in (7.5) and (7.6) are known as the *third type* and *fourth type*, respectively. Other types of problems can be solved similarly, and we leave these as exercises. In particular, each of the examples from Section III.4 has a counterpart here.

Our final objective of this chapter is to demonstrate that the weak solutions of certain of the preceding mixed initial-boundary value problems are necessarily strong or classical solutions. Specifically, we shall show that the weak solution is smooth for problems with smooth or regular data.

Consider the problem (7.4) above with $F \equiv 0$. The solution $u(\cdot)$ of the abstract problem is given by the semigroup constructed in Theorem 6.2 as $u(t) = T(t)u_0$. (We are assuming that $a(\cdot,\cdot)$ is V-elliptic.) Since $T(t) \in L(H, D(A))$ and $AT(t) \in \mathcal{L}(H)$ for all $t>0$, we obtain from the identity $(T(t/m))^m = T(t)$ that $T(t) \in L(H, D(A^m))$ for integer $m \ge 1$. This is an abstract regularity result; generally, for parabolic problems $D(A^m)$ consists of increasingly smooth functions as m gets large. Assume also that $a(\cdot,\cdot)$ is k-regular over V (cf. Section 6.4) for some integer $k \ge 0$. Then A^{-1} maps $H^s(G)$ into $H^{2+s}(G)$ for $0 \le s \le k$, so $D(A^m) \subset H^{2+k}$ whenever $2m \ge 2+k$. Thus, we have the spatial regularity result that $u(t) \in H^{2+k}(G)$ for all $t>0$

when $a(\cdot,\cdot)$ is V-elliptic and k-regular. One can clearly use the imbedding results of Section II.4 to show $U(\cdot,t) \in C_u^p(G)$ when $2(2+k) > 2p+n$.

We consider the regularity in time of the solution of the abstract problem corresponding to (7.4). First note that $A^m : D(A^m) \to H$ defines a scalar product on $D(A^m)$ for which $D(A^m)$ is a Hilbert space. Fix $t > \tau > 0$ and consider the identity

$$(1/h)(u(t+h) - u(t)) = A^{-m}[(1/h)(T(t+h-\tau) - T(t-\tau))A^m u(t)]$$

for $0 < |h| < t - \tau$. Since $A^m u(\tau) \in H$, the term in brackets converges in H, hence $u'(t) \in D(A^m)$ for all $t > 0$ and integer m. This is an abstract temporal regularity result. Assume now that $a(\cdot,\cdot)$ is k-regular over V. The preceding remarks show that the above difference quotients converge to $u'(t) = \partial_t U(\cdot,t)$ in the space $H^{2+k}(G)$. The convergence holds in $C_u^p(G)$ if $2(2+k) > 2p+n$ as before, and the solution U is a classical solution for $p \geq 2$. Thus, (7.4) has a classical solution when the above hypotheses hold for some $k > n/2$.

Exercises

1.1. Supply all details in Section 1.

1.2. Develop analogous series representations for the solution of (1.5) and (1.3) with the boundary conditions

(a) $u_x(0,t) = u_x(\pi,t) = 0$ of Neumann type (cf. Section III.7.7),

(b) $u(0,t) = u(\pi,t)$, $u_x(0,t) = u_x(\pi,t)$ of periodic type (cf. Section III.7.8).

1.3. Find the solution of the *backward heat equation*

$$u_t + u_{xx} = 0\ , \qquad 0 < x < \pi\ ,\ t > 0$$

subject to $u(0,t) = u(\pi,t) = 0$ and $u(x,0) = \sin(nx)/n$. Discuss the dependence of the solution on the initial data $u(x,0)$.

2.1. If A is given as in Section III.7.C, obtain the eigenfunction series representation for the solution of (2.1).

2.2. Show that if $u, v \in C^1((0,T), H)$, then

$$D_t(u(t), v(t))_H = (u'(t), v(t))_H + (u(t), v'(t))_H \ , \qquad 0 < t < T \ .$$

2.3. Show (2.3) holds for all $x \in H$.

2.4. Verify (2.5).

3.1. If $\{S(t)\}$ is a contraction semigroup with generator B, show that $\{e^{-\lambda t}S(t)\}$ is a contraction semigroup for $\lambda > 0$ and that its generator is $B - \lambda$.

3.2. Verify the limits as $t \to \infty$ in the two identities leading up to Theorem 3.1.

3.3. Show $B(\lambda - B)^{-1} = (\lambda - B)^{-1}B$ for B as in Theorem 3.1.

3.4. Show that Theorem 3.3 holds if we replace the given hypothesis on f by $f : \mathbb{R}^+ \to D(A)$ and $Af(\cdot) \in C([0,\infty), H)$.

4.1. Prove Lemma 4.1.

4.2. Show that the hypothesis in Theorem 4.3 that $D(A)$ is dense in H is unnecessary. Hint: If $x \in D(A)^\perp$, then $x = (\lambda + A)z$ for some $z \in D(A)$ and $z = \theta$.

4.3. Show that (4.1) follows from A being accretive.

4.4. For the operator A in Example 4(a), find the kernel and range of $\lambda + A$ for each $\lambda \geq 0$ and c with $|c| \leq 1$.

4.5. Solve (4.5) by the methods of Chapter III.

4.6. Solve (4.5) and (4.6) when the Dirichlet conditions are replaced by Neumann conditions. Repeat for other boundary conditions.

5.1. Show that operators $\{S_+(t)\}$ and $\{S_-(t)\}$ commute in the proof of Theorem 5.1.

5.2. Verify all details in the Example of Section 5.

5.3. If A is self-adjoint on the complex Hilbert space H, show iA generates a unitary group. Discuss the Cauchy problem for the Schrodinger equation $u_t = i\Delta_n u$ on $\mathbb{R}^n \times \mathbb{R}$.

5.4. Formulate and discuss some well-posed problems for the equation $\partial_t u + \partial_x^3 u = 0$ for $0 < x < 1$ and $t > 0$.

6.1. Verify all the estimates which lead to the convergence of the integral (6.8).

6.2. Finish the proof of Theorem 6.5.

6.3. Show that $f(t) \equiv \int_0^t F(s)\,ds$ is Hölder continuous if $F(\cdot) \in L^p(0,T;H)$ for some $p > 1$.

6.4. Show that for $0 < \varepsilon < \theta_0$ and integer $n \geq 1$, there is a constant $c_{\varepsilon,n}$ for which $\|t^n A^n T(t)\| \leq c_{\varepsilon,n}$ for $t \in S(\theta_0 - \varepsilon)$ in the situation of Theorem 6.2.

7.1. In the proof of Theorem 7.1(a), verify $\lim_{n\to\infty} \|U_n(\cdot,t) - u(t)\| = 0$. For Theorem 7.1(b), show $\varphi \in C_0^\infty(I, L^2(G))$.

7.2. Give a proof of Theorem 7.2 without appealing to the results of Corollary III.3.2.

7.3. Show that the change of variable $u(t) = e^{\lambda t} v(t)$ in (7.1) gives a corresponding equation with A replaced by $A + \lambda$. Verify that (7.4) is well-posed if $a_1(\cdot,\cdot)$ is strongly elliptic.

7.4. Show that (7.3) is equivalent to (7.5) for an appropriate choice of V. Show how to solve (7.5) with a non-homogeneous boundary condition.

7.5. Show that (7.3) is equivalent to (7.6) for an appropriate choice of V. Show how to solve (7.6) with a non-homogeneous boundary condition. If G is an interval, show *periodic boundary conditions* are obtained.

7.6. Solve initial-boundary value problems corresponding to each of examples in Sections 4.3, 4.4, and 4.5 of Chapter III.

7.7. Show that $u(t) = T(t)u_0$ converges to u_0 in $D(A^m)$ if and only if $u_0 \in D(A^m)$. Discuss the corresponding limit $\lim_{t\to 0^+} U(\cdot,t)$ in (7.4).

Chapter V

Implicit Evolution Equations

1 Introduction

We shall be concerned with evolution equations in which the time-derivative of the solution is not given explicitly. This occurs, for example, in problems containing the pseudoparabolic equation

$$\partial_t u(x,t) - a\partial_x^2 \partial_t u(x,t) - \partial_x^2 u(x,t) = f(x,t) \tag{1.1}$$

where the constant a is non-zero. However, (1.1) can be reduced to the standard evolution equation (3.4) in an appropriate space because the operator $I - a\partial_x^2$ which acts on $\partial_t u(x,t)$ can be inverted. Thus, (1.1) is an example of a *regular* equation; we study such problems in Section 2. Section 3 is concerned with those regular equations of a special form suggested by (1.1).

Another example which motivates some of our discussion is the partial differential equation

$$m(x)\partial_t u(x,t) - \partial_x^2 u(x,t) = f(x,t) \tag{1.2}$$

where the coefficient is non-negative at each point. The equation (1.2) is parabolic at those points where $m(x) > 0$ and elliptic where $m(x) = 0$. For such an equation of *mixed type* some care must be taken in order to prescribe a well posed problem. If $m(x) > 0$ almost everywhere, then (1.2) is a model of a regular evolution equation. Otherwise, it is a model of a *degenerate* equation. We study the Cauchy problem for degenerate equations in Section 4 and in Section 5 give more examples of this type.

2 Regular Equations

Let V_m be a Hilbert space with scalar-product $(\cdot,\cdot)_m$ and denote the corresponding Riesz map from V_m onto the dual V_m' by $\mathcal{M}$. That is,

$$\mathcal{M}x(y) = (x,y)_m \ , \qquad x,y \in V_m \ .$$

Let D be a subspace of V_m and $L : D \to V_m'$ a linear map. If $u_0 \in V_m$ and $f \in C((0,\infty), V_m')$ are given, we consider the problem of finding $u \in C([0,\infty), V_m) \cap C^1((0,\infty), V_m)$ such that

$$\mathcal{M}u'(t) + Lu(t) = f(t) \ , \qquad t > 0 \ , \tag{2.1}$$

and $u(0) = u_0$.

Note that (2.1) is a generalization of the evolution equation IV(2.1). If we identify V_m with V_m' by the Riesz map $\mathcal{M}$ (i.e., take $\mathcal{M} = I$) then (2.1) reduces to IV(2.1). In the general situation we shall solve (2.1) by reducing it to a Cauchy problem equivalent to IV(2.1).

We first obtain our a-priori estimate for a solution $u(\cdot)$ of (2.1), with $f = 0$ for simplicity. For such a solution we have

$$D_t(u(t), u(t))_m = -2\,\mathrm{Re}\, Lu(t)(u(t))$$

and this suggests consideration of the following.

Definition. The linear operator $L : D \to V_m'$ with $D \leq V_m$ is *monotone* (or *non-negative*) if

$$\mathrm{Re}\, Lx(x) \geq 0 \ , \qquad x \in D \ .$$

We call L *strictly monotone* (or *positive*) if

$$\mathrm{Re}\, Lx(x) > 0 \ , \qquad x \in D \ , \ x \neq 0 \ .$$

Our computation above shows there is at most one solution of the Cauchy problem for (2.1) whenever L is monotone, and it suggests that V_m is the correct space in which to seek well-posedness results for (2.1).

To obtain an (explicit) evolution equation in V_m which is equivalent to (2.1), we need only operate on (2.1) with the inverse of the isomorphism $\mathcal{M}$, and this gives

$$u'(t) + \mathcal{M}^{-1} \circ Lu(t) = \mathcal{M}^{-1} f(t) \ , \qquad t > 0 \ . \tag{2.2}$$

This suggests we define $A = \mathcal{M}^{-1} \circ L$ with domain $D(A) = D$, for then (2.2) is equivalent to IV(2.1). Furthermore, since $\mathcal{M}$ is the Riesz map determined by the scalar-product $(\cdot,\cdot)_m$, we have

$$(Ax, y)_m = Lx(y) \ , \qquad x \in D \ , \ y \in V_m \ . \tag{2.3}$$

This shows that L is monotone if and only if A is accretive. Thus, it follows from Theorem IV.4.3 that $-A$ generates a contraction semigroup on V_m if and only if L is monotone and $I + A$ is surjective. Since $\mathcal{M}(I + A) = \mathcal{M} + L$, we obtain the following result from Theorem IV.3.3.

Theorem 2.1 *Let $\mathcal{M}$ be the Riesz map of the Hilbert space V_m with scalar product $(\cdot,\cdot)_m$ and let L be linear from the subspace D of V_m into V_m'. Assume that L is monotone and $\mathcal{M} + L : D \to V_m'$ is surjective. Then, for every $f \in C^1([0,\infty), V_m')$ and $u_0 \in D$ there is a unique solution $u(\cdot)$ of (2.1) with $u(0) = u_0$.*

In order to obtain an analogue of the situation in Section IV.6, we suppose L is obtained from a continuous sesquilinear form. In particular, let V be a Hilbert space for which V is a dense subset of V_m and the injection is continuous; hence, we can identify $V_m' \subset V'$. Let $\ell(\cdot,\cdot)$ be continuous and sesquilinear on V and define the corresponding linear map $\mathcal{L} : V \to V'$ by

$$\mathcal{L}x(y) = \ell(x, y) \ , \qquad x, y \in V \ .$$

Define $D \equiv \{x \in V : \mathcal{L}x \in V_m'\}$ and $L = \mathcal{L}|_D$. Then (2.3) shows that

$$\ell(x, y) = (Ax, y)_m \ , \qquad x \in D \ , \ y \in V \ ,$$

so it follows that A is the operator determined by the triple $\{\ell(\cdot,\cdot), V, V_m\}$ as in Theorem IV.6.1. Thus, from Theorems IV.6.3 and IV.6.5 we obtain the following.

Theorem 2.2 *Let $\mathcal{M}$ be the Riesz map of the Hilbert space V_m with scalar-product $(\cdot,\cdot)_m$. Let $\ell(\cdot,\cdot)$ be a continuous, sesquilinear and elliptic form on the Hilbert space V, which is assumed dense and continuously imbedded in V_m, and denote the corresponding isomorphism of V onto V' by $\mathcal{L}$. Then for every Hölder continuous $f : [0,\infty) \to V_m'$ and $u_0 \in V_m$, there is a unique $u \in C([0,\infty), V_m) \cap C^1((0,\infty), V_m)$ such that $u(0) = u_0$, $\mathcal{L}u(t) \in V_m'$ for $t > 0$, and*

$$\mathcal{M}u'(t) + \mathcal{L}u(t) = f(t) \ , \qquad t > 0 \ . \tag{2.4}$$

We give four elementary examples to suggest the types of initial-boundary value problems to which the above results can be applied. In the first three of these examples we let $V_m = H_0^1(0,1)$ with the scalar-product

$$(u,v)_m = \int_0^1 (u\bar{v} + a\partial u \partial \bar{v}) \ ,$$

where $a > 0$.

2.1

Let $D = \{u \in H^2(0,1) \cap H_0^1(0,1) : u'(0) = cu'(1)\}$ where $|c| \le 1$, and define $LU = -\partial^3 u$. Then we have $Lu(\varphi) = (\partial^2 u, \partial\varphi)$ for $\varphi \in H_0^1(0,1)$, and (cf., Section IV.4)

$$2\,\mathrm{Re}\, Lu(u) = |u'(1)|^2 - |u'(0)|^2 \ge 0 \ , \qquad u \in D \ .$$

Thus, Theorem 2.1 shows that the initial-boundary value problem

$$\begin{aligned}
&(\partial_t - a\partial_x^2\partial_t)U(x,t) - \partial_x^3 U(x,t) = 0 \ , \qquad 0 < x < 1 \ , \ t \ge 0 \ ,\\
&U(0,t) = U(1,t) = 0 \ , \quad \partial_x U(0,t) = c\partial_x U(1,t) \ , \qquad t \ge 0 \ ,\\
&U(x,0) = U_0(x)
\end{aligned}$$

has a unique solution whenever $U_0 \in D$.

2.2

Let $V = H_0^2(0,1)$ and define

$$\ell(u,v) = \int_0^1 \partial^2 u \cdot \partial^2 \bar{v} \ , \qquad u,v \in V \ .$$

Then $D = H_0^2(0,1) \cap H^3(0,1)$ and $Lu = \partial^4 u$, $u \in D$. Theorem 2.2 then asserts the existence and uniqueness of a solution of the problem

$$\begin{aligned}
&(\partial_t - a\partial_x^2\partial_t)U(x,t) + \partial_x^4 U(x,t) = 0 \ , \qquad 0 < x < 1 \ , \ t > 0 \ ,\\
&U(0,t) = U(1,t) = \partial_x U(0,t) = \partial_x U(1,t) = 0 \ , \qquad t > 0 \ ,\\
&U(x,0) = U_0(x) \ , \qquad 0 < x < 1 \ ,
\end{aligned}$$

for each $U_0 \in H_0^1(0,1)$.

2.3

Let $V = H_0^1(0,1)$ and define

$$\ell(u,v) = \int_0^1 \partial u \partial \bar{v} \ , \qquad u, v \in V \ .$$

Then $D = V = V_m$ and $Lu = -\partial^2 u$, $u \in D$. From either Theorem 2.1 or 2.2 we obtain existence and uniqueness for the problem

$$(\partial_t - a\partial_x^2 \partial_t) U(x,t) - \partial_x^2 U(x,t) = 0 \ , \qquad 0 < x < 1 \ , \ t > 0 \ ,$$

$$U(0,t) = U(1,t) = 0 \ , \qquad t > 0 \ ,$$

$$U(x,0) = U_0(x) \ , \qquad 0 < x < 1 \ ,$$

whenever $U_0 \in D = V_m$.

2.4

For our last example we let V_m be the completion of $C_0^\infty(G)$ with the scalar-product

$$(u,v)_m \equiv \int_G m(x) u(x) \overline{v(x)} \, dx \ .$$

We assume G is open in $\mathbb{R}^n$ and $m \in L^\infty(G)$ is given with $m(x) > 0$ for a.e. $x \in G$. (Thus, V_m is the set of measurable functions u on G for which $m^{1/2} u \in L^2(G)$.) Let $V = H_0^1(G)$ and define

$$\ell(u,v) = \int_G \nabla u \cdot \nabla \bar{v} \ , \qquad u, v \in V \ .$$

Then Theorem 2.2 implies the existence and uniqueness of a solution of the problem

$$m(x) \partial_t U(x,t) - \Delta_n U(x,t) = 0 \ , \qquad x \in G \ , \ t > 0 \ ,$$

$$U(s,t) = 0 \ , \qquad s \in \partial G \ , \ t > 0 \ ,$$

$$U(x,0) = U_0(x) \ , \qquad x \in G \ .$$

Note that the initial condition is attained in the sense that

$$\lim_{t \to 0^+} \int_G m(x) |U(x,t) - U_0(x)|^2 \, dx = 0 \ .$$

The first two of the preceding examples illustrate the use of Theorems 2.1 and 2.2 when $\mathcal{M}$ and L are both differential operators with the order of L strictly higher than the order of M. The equation in (2.2) is called *metaparabolic* and arises in special models of diffusion or fluid flow. The equation in (2.3) arises similarly and is called *pseudoparabolic*. We shall discuss this class of problems in Section 3. The last example (2.4) contains a *weakly degenerate* parabolic equation. We shall study such problems in Section 4 where we shall assume only that $m(x) \geq 0$, $x \in G$. This allows the equation to be of *mixed type*: parabolic where $m(x) > 0$ and elliptic where $m(x) = 0$. Such examples will be given in Section 5.

3 Pseudoparabolic Equations

We shall consider some evolution equations which generalize the example (2.3). Two types of solutions will be discussed, and we shall show how these two types differ essentially by the boundary conditions they satisfy.

Theorem 3.1 *Let V be a Hilbert space, suppose $m(\cdot,\cdot)$ and $\ell(\cdot,\cdot)$ are continuous sesquilinear forms on V, and denote by $\mathcal{M}$ and $\mathcal{L}$ the corresponding operators in $\mathcal{L}(V,V')$. (That is, $\mathcal{M}x(y) = m(x,y)$ and $\mathcal{L}x(y) = \ell(x,y)$ for $x, y \in V$.) Assume that $m(\cdot,\cdot)$ is V-coercive. Then for every $u_0 \in V$ and $f \in C(\mathbb{R}, V')$, there is a unique $u \in C^1(\mathbb{R}, V)$ for which (2.4) holds for all $t \in \mathbb{R}$ and $u(0) = u_0$.*

Proof: The coerciveness assumption shows that $\mathcal{M}$ is an isomorphism of V onto V', so the operator $A \equiv \mathcal{M}^{-1} \circ \mathcal{L}$ belongs to $\mathcal{L}(V)$. We can define $\exp(-tA) \in \mathcal{L}(V)$ as in Theorem IV.2.1 and then define

$$u(t) = \exp(-tA) \cdot u_0 + \int_0^t \exp(A(\tau - t)) \circ \mathcal{M}^{-1} f(\tau)\, d\tau \ , \qquad t > 0 \ . \quad (3.1)$$

Since the integrand is continuous and appropriately bounded, it follows that (3.1) is a solution of (2.2), hence of (2.1). We leave the proof of uniqueness as an exercise.

We call the solution $u(\cdot)$ given by Theorem 3.1 a *weak solution* of (2.1). Suppose we are given a Hilbert space H in which V is a dense subset, continuously imbedded. Thus $H \subset V'$ and we can define $D(M) = \{v \in V : \mathcal{M}v \in H\}$, $D(L) = \{v \in V : \mathcal{L}v \in H\}$ and corresponding operators $M = \mathcal{M}|_{D(M)}$

and $L = \mathcal{L}|_{D(L)}$ in H. A solution $u(\cdot)$ of (2.1) for which each term in (2.1) belongs to $C(\mathbb{R}, H)$ (instead of $C(\mathbb{R}, V')$) is called a *strong solution*. Such a function satisfies

$$Mu'(t) + Lu(t) = f(t) \ , \qquad t \in \mathbb{R} \ . \tag{3.2}$$

Theorem 3.2 *Let the Hilbert space V and operators $\mathcal{M}, \mathcal{L} \in \mathcal{L}(V, V')$ be given as in Theorem 3.1. Let the Hilbert space H be given as above and define the domains $D(M)$ and $D(L)$ and operators M and L as above. Assume $D(M) \subset D(L)$. Then for every $u_0 \in D(M)$ and $f \in C(\mathbb{R}, H)$ there is a (unique) strong solution $u(\cdot)$ of (3.2) with $u(0) = u_0$.*

Proof: By making the change-of-variable $v(t) = e^{-\lambda t}u(t)$ for some $\lambda > 0$ sufficiently large, we may assume without loss of generality that $D(M) = D(L)$ and $\ell(\cdot,\cdot)$ is V-coercive. Then L is a bijection onto H so we can define a norm on $D(L)$ by $\|v\|_{D(L)} = \|Lv\|_H$, $v \in D(L)$, which makes $D(L)$ a Banach space. (Clearly, $D(L)$ is also a Hilbert space.) Since $\ell(\cdot,\cdot)$ is V-coercive, it follows that for some $c > 0$

$$c\|v\|_V^2 \le \|Lv\|_H \|v\|_H \ , \qquad v \in D(L) \ ,$$

and the continuity of the injection $V \hookrightarrow H$ shows then that the injection $D(L) \hookrightarrow V$ is continuous. The operator $A \equiv \mathcal{M}^{-1}\mathcal{L} \in \mathcal{L}(V)$ leaves invariant the subspace $D(L)$. This implies that the restriction of A to $D(L)$ is a closed operator in the $D(L)$-norm. To see this, note that if $v_n \in D(L)$ and if $\|v_n - u_0\|_{D(L)} \to 0$, $\|Av_n - v_0\|_{D(L)} \to 0$, then

$$\begin{aligned} \|v_0 - Au_0\|_V &\le \|v_0 - Av_n\|_V + \|A(v_n - u_0)\|_V \\ &\le \|v_0 - Av_n\|_V + \|A\|_{\mathcal{L}(V)} \|v_n - u_0\|_V \ , \end{aligned}$$

so the continuity of $D(L) \hookrightarrow V$ implies that each of these terms converges to zero. Hence, $v_0 = Au_0$.

Since $A|_{D(L)}$ is closed and defined everywhere on $D(L)$, it follows from Theorem III.7.5 that it is continuous on $D(L)$. Therefore, the restrictions of the operators $\exp(-tA)$, $t \in \mathbb{R}$, are continuous on $D(L)$, and the formula (3.1) in $D(L)$ gives a strong solution as desired.

Corollary 3.3 *In the situation of Theorem 3.2, the weak solution $u(\cdot)$ is a strong solution if and only if $u_0 \in D(M)$.*

3.1

We consider now an abstract *pseudoparabolic* initial-boundary value problem. Suppose we are given the Hilbert spaces, forms and operators as in Theorem IV.7.2. Let $\varepsilon > 0$ and define

$$\begin{aligned} m(u,v) &= (u,v)_H + \varepsilon a(u,v) \\ \ell(u,v) &= a(u,v) \ , \qquad\qquad u,v \in V \ . \end{aligned}$$

Thus, $D(M) = D(L) = D(A)$. Let $f \in C(\mathbb{R}, H)$. If $u(\cdot)$ is a strong solution of (3.2), then we have

$$\left.\begin{aligned} & u'(t) + \varepsilon A_1 u'(t) + A_1 u(t) = f(t) \ , \\ & u(t) \in V \ , \quad \text{and} \\ & \partial_1 u(t) + \mathcal{A}_2 \gamma(u(t)) = 0 \ , \qquad\qquad t \in \mathbb{R} \ . \end{aligned}\right\} \tag{3.3}$$

Suppose instead that $F \in C(\mathbb{R}, H)$ and $g \in C(\mathbb{R}, B')$. If we define

$$f(t)(v) \equiv (F(t), v)_H + g(t)(\gamma(v)) \ , \qquad v \in V \ , \ t \in \mathbb{R} \ .$$

then a weak solution $u(\cdot)$ of (2.4) can be shown by a computation similar to the proof of Theorem III.3.1 to satisfy

$$\left.\begin{aligned} & u'(t) + \varepsilon A_1 u'(t) + A_1 u(t) = F(t) \ , \\ & u(t) \in V \ , \quad \text{and} \\ & \partial_1(\varepsilon u'(t) + u(t)) + \mathcal{A}_2(\gamma(\varepsilon u'(t) + u(t))) = g(t) \ , \qquad t \in \mathbb{R} \ . \end{aligned}\right\} \tag{3.4}$$

Note that (3.3) implies more than (3.4) with $g \equiv 0$. By taking suitable choices of the operators above, we could obtain examples of initial-boundary value problems from (3.3) and (3.4) as in Theorem IV.7.3.

3.2

For our second example we let G be open in $\mathbb{R}^n$ and choose $V = \{v \in H^1(G) : v(s) = 0. \text{ a.e. } s \in \Gamma\}$, where Γ is a closed subset of ∂G. We define

$$m(u,v) = \int_G \nabla u(x) \cdot \nabla \overline{v(x)} \, dx \ , \qquad u, v \in V$$

and assume $m(\cdot,\cdot)$ is V-elliptic. (Sufficient conditions for this situation are given in Corollary III.5.4.) Choose $H = L^2(G)$ and $V_0 = H_0^1(G)$; the corresponding partial differential operator $M : V \to V_0' \le \mathcal{D}^*(G)$ is given by

$Mu = -\Delta_n u$, the Laplacian (cf. Section III.2.2). Thus, from Corollary III.3.2 it follows that $D(M) = \{u \in V : \Delta_n u \in L^2(G),\ \partial u = 0\}$ where ∂ is the normal derivative ∂_ν on $\partial G \sim \Gamma$ whenever ∂G is sufficiently smooth. (Cf. Section III.2.3.) Define a second form on V by

$$\ell(u,v) = \int_G a(x)\partial_n u(x)\overline{v(x)}\,dx\ , \qquad u, v \in V\ ,$$

and note that $L = \mathcal{L} : V \to H \leq V'$ is given by $\mathcal{L}u = a(x)(\partial u/\partial x_n)$, where $a(\cdot) \in L^\infty(G)$ is given. Assume that for each $t \in \mathbb{R}$ we are given $F(\cdot,t) \in L^2(G)$ and that the map $t \mapsto F(\cdot,t) : \mathbb{R} \to L^2(G)$ is continuous. Let $g(\cdot,t) \in L^2(\partial G)$ be given similarly, and define $f \in C(\mathbb{R}, V')$ by

$$f(t)(v) = \int_G F(x,t)\overline{v(x)}\,dx + \int_{\partial G} g(s,t)\overline{v(s)}\,ds\ , \qquad t \in \mathbb{R}\ ,\ v \in V\ .$$

If $u_0 \in V$, then Theorem 3.1 gives a unique weak solution $u(\cdot)$ of (2.4) with $u(0) = u_0$. That is

$$m(u'(t),v) + \ell(u(t),v) = f(t)(v)\ , \qquad v \in V\ ,\ t \in \mathbb{R}\ ,$$

and this is equivalent to

$$\begin{aligned} &Mu'(t) + Lu(t) = F(\cdot,t)\ , && t \in \mathbb{R} \\ &u(t) \in V\ , \quad \partial_t(\partial u(t)) = g(\cdot,t)\ . && \end{aligned}$$

From Theorem IV.7.1 we thereby obtain a generalized solution $U(\cdot,\cdot)$ of the initial-boundary value problem

$$\begin{aligned} &-\Delta_n\partial_t U(x,t) + a(x)\partial_n U(x,t) = F(x,t)\ , && x \in G\ ,\ t \in \mathbb{R}\ , \\ &U(s,t) = 0\ , && s \in \Gamma\ , \\ &\partial_\nu U(s,t) = \partial_\nu U_0(s) + \int_0^t g(s,\tau)\,d\tau\ , && s \in \partial G \sim \Gamma\ , \\ &U(x,0) = U_0(x)\ , && x \in G\ . \end{aligned}$$

Finally, we note that $f \in C(\mathbb{R}, H)$ if and only if $g \equiv 0$, and then $\partial_\nu U(s,t) = \partial_\nu U_0(s)$ for $s \in \partial G \sim \Gamma$, $t \in \mathbb{R}$; thus, $U(\cdot,t) \in D(M)$ if and only if $U_0 \in D(M)$. This agrees with Corollary 3.3.

4 Degenerate Equations

We shall consider the evolution equation (2.1) in the situation where $\mathcal{M}$ is permitted to degenerate, i.e., it may vanish on non-zero vectors. Although it is not possible to rewrite it in the form (2.2), we shall essentially factor the equation (2.1) by the kernel of $\mathcal{M}$ and thereby obtain an equivalent problem which is regular.

Let V be a linear space and $m(\cdot,\cdot)$ a sesquilinear form on V that is symmetric and non-negative:

$$m(x,y) = \overline{m(x,y)} \;, \qquad x, y \in V \;,$$
$$m(x,x) \geq 0 \;, \qquad x \in V \;.$$

Then it follows that

$$|m(x,y)|^2 \leq m(x,x) \cdot m(y,y) \;, \qquad x, y \in V \;, \tag{4.1}$$

and that $x \mapsto m(x,x)^{1/2} = \|x\|_m$ is a seminorm on V. Let V_m denote this seminorm space whose dual V_m' is a Hilbert space (cf. Theorem I.3.5). The identity

$$\mathcal{M}x(y) = m(x,y) \;, \qquad x, y \in V$$

defines $\mathcal{M} \in \mathcal{L}(V_m, V_m')$ and it is just such an operator which we shall place in the leading term in our evolution equation. Let $D \leq V$, $L \in L(D, V_m')$, $f \in C((0,\infty), V_m')$ and $g_0 \in V_m'$. We consider the problem of finding a function $u(\cdot) : [0,\infty) \to V$ such that

$$\mathcal{M}u(\cdot) \in C([0,\infty), V_m') \cap C^1((0,\infty), V_m') \;, \qquad (\mathcal{M}u)(0) = g_0 \;,$$

and $u(t) \in D$ with

$$(\mathcal{M}u)'(t) + Lu(t) = f(t) \;, \qquad t > 0 \;. \tag{4.2}$$

(Note that when $m(\cdot,\cdot)$ is a scalar product on V_m and V_m is complete then $\mathcal{M}$ is the Riesz map and (4.2) is equivalent to (2.1).)

Let K be the kernel of the linear map $\mathcal{M}$ and denote the corresponding quotient space by V/K. If $q : V \to V/K$ is the canonical surjection, then we define by

$$m_0(q(x), q(y)) = m(x,y) \;, \qquad x, y \in V$$

a scalar product $m_0(\cdot,\cdot)$ on V/K. The completion of V/K, $m_0(\cdot,\cdot)$ is a Hilbert space W whose scalar product is also denoted by $m_0(\cdot,\cdot)$. (Cf. Theorem I.4.2.) We regard q as a map of V_m into W; thus, it is norm-preserving and has a dense range, so its dual $q' : W' \to V_m'$ is a norm-preserving isomorphism (Corollary I.5.3) defined by

$$q'(f)(x) = f(q(x)) \ , \qquad f \in W' \ , \ x \in V_m \ .$$

If $\mathcal{M}_0$ denotes the Riesz map of W with the scalar product $m_0(\cdot,\cdot)$, then we have

$$\begin{aligned} q'\mathcal{M}_0 q(x)(y) &= \mathcal{M}_0 q(x)(q(y)) = m_0(q(x), q(y)) \\ &= \mathcal{M}x(y) \ , \end{aligned}$$

hence,

$$q'\mathcal{M}_0 q = \mathcal{M} \ . \tag{4.3}$$

From the linear map $L : D \to V_m'$ we want to construct a linear map L_0 on the image $q[D]$ of $D \leq V_m$ by q so that it satisfies

$$q' L_0 q = L \ . \tag{4.4}$$

This is possible if (and, in general, only if) $K \cap D$ is a subspace of the kernel of L, $K(L)$ by Theorem I.1.1, and we shall assume this is so.

Let $f(\cdot)$ and g_0 be given as above and consider the problem of finding a function $v(\cdot) \in C([0,\infty), W) \cap C^1((0,\infty), W)$ such that $v(0) = (q'\mathcal{M}_0)^{-1} g_0$ and

$$\mathcal{M}_0 v'(t) + L_0 v(t) = (q')^{-1} f(t) \ , \qquad t > 0 \ . \tag{4.5}$$

Since the domain of L_0 is $q[D]$, if $v(\cdot)$ is a solution of (4.5) then for each $t > 0$ we can find a $u(t) \in D$ for which $v(t) = q(u(t))$. But $q'\mathcal{M}_0 : W \to V_m'$ is an isomorphism and so from (4.3), (4.4) and (4.5) it follows that $u(\cdot)$ is a solution of (4.2) with $\mathcal{M}u(0) = g_0$. This leads to the following results.

Theorem 4.1 *Let V_m be a seminorm space obtained from a symmetric and non-negative sesquilinear form $m(\cdot,\cdot)$, and let $\mathcal{M} \in \mathcal{L}(V_m, V_m')$ be the corresponding linear operator given by $\mathcal{M}x(y) = m(x,y)$, $x, y \in V_m$. Let D be a subspace of V_m and $L : D \to V_m'$ be linear and monotone.* (a) *If $K(\mathcal{M}) \cap D \leq K(L)$ and if $\mathcal{M} + L : D \to V_m'$ is a surjection, then for every $f \in C^1([0,\infty), V_m')$ and $u_0 \in D$ there exists a solution of* (4.2) *with $(\mathcal{M}u)(0) = \mathcal{M}u_0$.* (b) *If $K(\mathcal{M}) \cap K(L) = \{0\}$, then there is at most one solution.*

Proof: The existence of a solution will follow from Theorem 2.1 applied to (4.5) if we show $L_0 : q[D] \to W'$ is monotone and $\mathcal{M}_0 + L_0$ is onto. But (4.5) shows L_0 is monotone, and the identity

$$q'(\mathcal{M}_0 + L_0)q(x) = (\mathcal{M} + L)(x) \ , \qquad x \in D \ ,$$

implies that $\mathcal{M}_0 + L_0$ is surjective whenever $\mathcal{M} + L$ is surjective.

To establish the uniqueness result, let $u(\cdot)$ be a solution of (4.2) with $f \equiv 0$ and $\mathcal{M}u(0) = 0$; define $v(t) = qu(t)$, $t \geq 0$. Then

$$D_t m_0(v(t), v(t)) = 2\,\mathrm{Re}(\mathcal{M}_0 v'(t))(v(t)) \ , \qquad t > 0 \ ,$$

and this implies by (4.3) that

$$\begin{aligned} D_t m(u(t), u(t)) &= 2\,\mathrm{Re}(\mathcal{M}u)'(t)(u(t)) \\ &= -2\,\mathrm{Re}\, Lu(t)(u(t)) \ , \qquad t > 0 \ . \end{aligned}$$

Since L is monotone, this shows $\mathcal{M}u(t) = 0$, $t \geq 0$, and (4.2) implies $Lu(t) = 0$, $t > 0$. Thus $u(t) \in K(\mathcal{M}) \cap K(L)$, $t \geq 0$, and the desired result follows.

We leave the proof of the following analogue of Theorem 2.2 as an exercise.

Theorem 4.2 *Let V_m be a seminorm space obtained from a symmetric and non-negative sesquilinear form $m(\cdot,\cdot)$, and let $\mathcal{M} \in \mathcal{L}(V_m, V_m')$ denote the corresponding operator. Let V be a Hilbert space which is dense and continuously imbedded in V_m. Let $\ell(\cdot,\cdot)$ be a continuous, sesquilinear and elliptic form on V, and denote the corresponding isomorphism of V onto V' by $\mathcal{L}$. Let $D = \{u \in V : \mathcal{L}u \in V_m'\}$. Then, for every Hölder continuous $f : [0,\infty) \to V_m'$ and every $u_0 \in V_m$, there exists a unique solution of* (4.2) *with* $(\mathcal{M}u)(0) = \mathcal{M}u_0$.

5 Examples

We shall illustrate the applications of Theorems 4.1 and 4.2 by solving some initial-boundary value problems with partial differential equations of mixed type.

5.1

Let $V_m = L^2(0,1)$, $0 \le a < b \le 1$, and

$$m(u,v) = \int_a^b u(x)\overline{v(x)}\,dx\ , \qquad v \in V_m\ .$$

Then $V_m' = L^2(a,b)$, which we identify as that subspace of $L^2(0,1)$ whose elements are zero a.e. on $(0,a)\cup(b,1)$, and $\mathcal{M}$ becomes multiplication by the characteristic function of the interval (a,b). Let $L = \partial$ with domain $D = \{u \in H^1(0,1) : u(0) = cu(1),\ \partial u \in V_m' \subset L^2(0,1)\}$. We assume $|c| \le 1$, so L is monotone (cf. Section IV.4(a)). Note that each function in D is constant on $(0,a)\cup(b,1)$. Thus, $K(\mathcal{M})\cap D = \{0\}$ and $K(\mathcal{M})\cap D \le K(L)$ follows. Also, note that $K(L)$ is either $\{0\}$ or consists of the constant functions, depending on whether or not $c \ne 1$, respectively. Thus, $K(\mathcal{M}) \cap K(L) = \{0\}$. If u is the solution of (cf. Section IV.4(a))

$$u(x) + \partial u(x) = f(x)\ , \qquad a < x < b\ ,\ u(a) = cu(b)$$

and is extended to $(0,1)$ by being constant on each of the intervals, $[0,a]$ and $[b,1]$, then $(\mathcal{M}+L)u = f \in V_m'$. Hence $\mathcal{M}+L$ maps onto V_m' and Theorem 4.1 asserts the existence and uniqueness of a generalized solution of the problem

$$\left.\begin{aligned}
&\partial_t U(x,t) + \partial_x U(x,t) = F(x,t)\ , \qquad a < x < b\ ,\ t \ge 0\ ,\\
&\partial_x U(x,t) = 0\ , \qquad x \in (0,a)\cup(b,a)\ ,\\
&U(0,t) = cU(1,t)\ , \qquad U(x,0) = U_0(x)\ , \qquad a < x < b\ ,
\end{aligned}\right\} \tag{5.1}$$

for appropriate $F(\cdot,\cdot)$ and U_0. This example is trivial (i.e., equivalent to Section IV.4(a) on the interval (a,b)) but motivates the proof-techniques of Section 4.

5.2

We consider some problems with a partial differential equation of mixed elliptic-parabolic type. Let $m_0(\cdot) \in L^\infty(G)$ with $m_0(x) \ge 0$, a.e. $x \in G$, an open subset of $\mathbb{R}^n$ whose boundary ∂G is a C^1-manifold with G on one side of ∂G. Let $V_m = L^2(G)$ and

$$m(u,v) = \int_G m_0(x)u(x)\overline{v(x)}\,dx\ , \qquad u,v \in V_m\ .$$

Then $\mathcal{M}$ is multiplication by $m_0(\cdot)$ and maps $L^2(G)$ into $V'_m \equiv \{\sqrt{m_0}\cdot g : g \in L^2(G)\} \subset L^2(G)$. Let Γ be a closed subset of ∂G and define $V = \{v \in H^1(G) : \gamma_0 v = 0 \text{ on } \Gamma\}$ as in Section III.4.1. Let

$$\ell(u,v) = \int_G \nabla u \cdot \overline{\nabla v}\, dx \ , \qquad u, v \in V \tag{5.2}$$

and assume $\sum \equiv \{s \in \partial G : \nu_n(s) > 0\} \subset \Gamma$. Thus, Theorem III.5.3 implies $\ell(\cdot,\cdot)$ is V-elliptic, so $\mathcal{M} + \mathcal{L}$ maps onto V', hence, onto V'_m. Theorem 4.2 shows that if $U_0 \in L^2(G)$ and if F is given as in Theorem IV.7.3, then there is a unique generalized solution of the problem

$$\left.\begin{aligned}
&\partial_t(m_0(x)U(x,t)) - \Delta_n U(x,t) = m_0(x)F(x,t) \ , && x \in G \ , \\
&U(s,t) = 0 \ , \qquad s \in \Gamma \ , && \\
&\frac{\partial U(s,t)}{\partial \nu} = 0 \ , \quad s \in \partial G \sim \Gamma \ , && t > 0 \ , \\
&m_0(x)(U(x,0) - U_0(x)) = 0 \ . &&
\end{aligned}\right\} \tag{5.3}$$

The partial differential equation in (5.3) is parabolic at those $x \in G$ for which $m_0(x) > 0$ and elliptic where $m_0(x) = 0$. The boundary conditions are of mixed Dirichlet-Neumann type (cf. Section III.4.1) and the initial value of $U(x,0)$ is prescribed only at those points of G at which the equation is parabolic.

Boundary conditions of the third type may be introduced by modifying $\ell(\cdot,\cdot)$ as in Section III.4.2. Similarly, by choosing

$$\ell(u,v) = \int_G \nabla u \cdot \overline{\nabla v}\, dx + (\gamma_0 u)\overline{(\gamma_0 v)}$$

on $V = \{u \in H^1(G) : \gamma_0 u \text{ is constant}\}$, we obtain a unique generalized solution of the initial-boundary value problem of *fourth type* (cf., Section III.4.2)

$$\left.\begin{aligned}
&\partial_t(m_0(x)U(x,t)) - \Delta_n U(x,t) = m_0(x)F(x,t) \ , && x \in G \ , \\
&U(s,t) = h(t) \ , && s \in \partial G \ , \\
&\left(\int_{\partial G} \frac{\partial U(s,t)}{\partial \nu}\, ds \Big/ \int_{\partial G} ds\right) + h(t) = 0 \ , && t > 0 \ , \\
&m_0(x)(U(x,0) - U_0(x)) = 0 \ . &&
\end{aligned}\right\} \tag{5.4}$$

The data $F(\cdot,\cdot)$ and U_0 are specified as before; $h(\cdot)$ is unknown and part of the problem.

5.3

Problems with a partial differential equation of mixed pseudoparabolic-parabolic type can be similarly handled. Let $m_0(\cdot)$ be given as above and define

$$m(u,v) = \int_G (u(x)\overline{v(x)} + m_0(x)\nabla u(x) \cdot \overline{\nabla v}\,(x))\,dx\ , \qquad u,v \in V_m\ ,$$

with $V_m = H^1(G)$. Then $V_m \hookrightarrow L^2(G)$ is continuous so we can identify $L^2(G) \leq V_m'$. Define $\ell(\cdot,\cdot)$ by (5.2) where V is a subspace of $H^1(G)$ which contains $C_0^\infty(G)$ and is to be prescribed. Then $K(\mathcal{M}) = \{0\}$ and $m(\cdot,\cdot) + \ell(\cdot,\cdot)$ is V-coercive, so Theorem 4.2 will apply. In particular, if $U_0 \in L^2(G)$ and F as in Theorem IV.7.3 are given, then there is a unique solution of the equation

$$\partial_t(U(x,t) - \sum_{j=1}^{n} \partial_j(m_0(x)\partial_j U(x,t))) - \Delta_n U(x,t) = F(x,t)\ , \qquad x \in G\ ,\ t > 0\ ,$$

with the initial condition

$$U(x,0) = U_0(x)\ , \qquad x \in G\ ,$$

and boundary conditions which depend on our choice of V.

5.4

We consider a problem with a time derivative and possibly a partial differential equation on a boundary. Let G be as in (5.2) and assume for simplicity that ∂G intersects the hyperplane $\mathbb{R}^{n-1} \times \{0\}$ in a set with relative interior S. Let $a_n(\cdot)$ and $b(\cdot)$ be given nonnegative, real-valued functions in $L^\infty(S)$. We define $V_m = H^1(G)$ and

$$m(u,v) = \int_G u(x)\overline{v(x)}\,dx + \int_S a(s)u(s)\overline{v(s)}\,ds\ , \qquad u,v \in V_m\ ,$$

where we suppress the notation for the trace operator, i.e., $u(s) = (\gamma_0 u)(s)$ for $s \in \partial G$. Define V to be the completion of $C^\infty(\bar{G})$ with the norm given by

$$\|v\|_V^2 \equiv \|v\|_{H^1(G)}^2 + \left(\int_S b(s) \sum_{j=1}^{n-1} |D_j v(s)|^2\,ds\right)\ .$$

Thus, V consists of these $v \in H^1(G)$ for which $b^{1/2} \cdot \partial_j(\gamma_0 v) \in L^2(S)$ for $1 \leq j \leq n-1$; it is a Hilbert space. We define

$$\ell(u,v) = \int_G \nabla u(x) \cdot \nabla \overline{v(x)}\, dx + \int_S b(s)\Big(\sum_{j=1}^{n-1} \partial_j u(s) \partial_j \overline{v(s)}\Big)\, ds\ , \qquad u, v \in V\ .$$

Then $K(\mathcal{M}) = \{0\}$ and $m(\cdot,\cdot)+\ell(\cdot,\cdot)$ is V-coercive. If $U_0 \in L^2(G)$ and $F(\cdot,\cdot)$ is given as above, then Theorem 4.2 asserts the existence and uniqueness of the solution $U(\cdot,\cdot)$ of the initial-boundary value problem

$$\begin{cases} \partial_t U(x,t) - \Delta_n U(x,t) = F(x,t)\ , & x \in G\ ,\ t > 0\ , \\ \partial_t(a(s)U(s,t)) + \dfrac{\partial U(s,t)}{\partial \nu} = \displaystyle\sum_{j=1}^{n-1} \partial_j (b(s) \partial_j U(s,t))\ , & s \in S\ , \\ \dfrac{\partial U(s,t)}{\partial \nu} = 0\ , & s \in \partial G \sim S\ , \\ b(s) \dfrac{\partial U(s,t)}{\partial \nu_S} = 0\ , & s \in \partial S\ , \\ U(x,0) = U_0(x)\ , & x \in G\ , \\ a(s)(U(s,0) - U_0(s)) = 0\ , & s \in S\ . \end{cases}$$

Similar problems with a partial differential equation of mixed type or other combinations of boundary conditions can be handled by the same technique. Also, the $(n-1)$-dimensional surface S can occur inside the region G as well as on the boundary. (Cf., Section III.4.5.)

Exercises

1.1. Use the separation-of-variables technique to obtain a series representation for the solution of (1.1) with $u(0,t) = u(\pi,t) = 0$ and $u(x,0) = u_0(x)$, $0 < x < \pi$. Compare the rate of convergence of this series with that of Section IV.1.

2.1. Provide all details in support of the claim that Theorem 2.1 follows from Theorem IV.3.3. Show that Theorem 2.2 follows from Theorems IV.6.3 and IV.6.5.

2.2. Show that Theorem 2.2 remains true if we replace the hypothesis that $\mathcal{L}$ is V-elliptic by $\lambda\mathcal{M} + \mathcal{L}$ is V-elliptic for some $\lambda \in \mathbb{R}$.

2.3. Characterize V'_m in each of the examples (2.1)–(2.3). Construct appropriate terms for $f(t)$ in Theorems 2.1 and 2.2. Write out the corresponding initial-boundary value problems that are solved.

2.4. Show $V'_m = \{m^{1/2}v : v \in L^2(0,1)\}$ in (2.4). Describe appropriate non-homogeneous terms for the partial differential equation in (2.4).

3.1. Verify that (3.1) is a solution of (2.2) in the situation of Theorem 3.1.

3.2. Prove uniqueness holds in Theorem 3.1. [Hint: Show $\sigma(t) \equiv \|u(t)\|_V^2$ satisfies $|\sigma'(t)| \leq K\sigma(t)$, $t \in \mathbb{R}$, where u is a solution of the homogeneous equation, then show that $\sigma(t) \leq \exp(K|t|) \cdot \sigma(0)$.]

3.3. Verify that (3.4) characterizes the solution of (2.4) in the case of Section 3.1. Discuss the regularity of the solution when $a(\cdot, \cdot)$ is k-regular.

4.1. Prove (4.1).

4.2. Prove Theorem 4.2. [Hint: Compare with Theorem 2.2.]

5.1. Give sufficient conditions on the data F, u_0 in (5.1) in order to apply Theorem 4.1.

5.2. Extend the discussion in Section 5.2 to include boundary conditions of the third type.

5.3. Characterize V'_m in Section 5.3. Write out the initial-boundary value problem solved in Section 5.3 for several choices of V.

5.4. Write out the problem solved in Section 5.4 when S is an interface as in Section III.4.5.

Chapter VI

Second Order Evolution Equations

1 Introduction

We shall find well-posed problems for evolution equations which contain the second order time derivative of the solution. These arise, for example, when we attempt to use the techniques of the preceding chapters to solve a Cauchy problem for the *wave equation*

$$\partial_t^2 u(x,t) - \Delta_n u(x,t) = F(x,t) \ . \tag{1.1}$$

The corresponding abstract problem will contain the second order evolution equation

$$u''(t) + \mathcal{A}u(t) = f(t) \ , \tag{1.2}$$

where $\mathcal{A}$ is an operator which contains $-\Delta_n$ in some sense. Wave equations with damping or friction occur in practice, e.g., the *telegraphists equation*

$$\partial_t^2 u(x,t) + R \cdot \partial_t u(x,t) - \Delta_n u(x,t) = F(x,t) \ ,$$

so we shall add terms to (1.2) of the form $\mathcal{B}u'(t)$. Finally, certain models in fluid mechanics lead to equations, for example,

$$\partial_t^2 (\Delta_n u(x,t)) + \partial_n^2 u(x,t) = 0 \ , \qquad x = (x_1, \ldots, x_n) \ , \tag{1.3}$$

which contain spatial derivatives in the terms with highest (= second) order time derivatives. These motivate us to consider abstract evolution equations of the form

$$\mathcal{C}u''(t) + \mathcal{B}u'(t) + \mathcal{A}u(t) = f(t) \ , \qquad t > 0 \ . \tag{1.4}$$

We consider in Section 2 equations of the form (1.4) in which $\mathcal{C}$ is invertible; this situation is similar to that of Section V.2, so we call (1.4) a *regular equation* then. The equation (1.3) is known as Sobolev's equation, so we call (1.4) a *Sobolev equation* when $\mathcal{C}$ is invertible and both $\mathcal{C}^{-1}\mathcal{B}$ and $\mathcal{C}^{-1}\mathcal{A}$ are continuous. This situation is studied in Section 3 and is the analogue of (first-order) pseudoparabolic problems. Section 4 will be concerned with (1.4) when $\mathcal{C}$ is *degenerate* in the sense of Section V.4. Such equations arise, for example from a system described by appropriately coupled wave and heat equations

$$\partial_t^2 u(x,t) - \Delta_n u(x,t) = 0 \ , \qquad x \in G_1 \ ,$$
$$\partial_t u(x,t) - \Delta_n u(x,t) = 0 \ , \qquad x \in G_2 \ .$$

Here the operator $\mathcal{C}$ is multiplication by the characteristic function of G_1 and $\mathcal{B}$ is multiplication by the characteristic function of G_2. G_1 and G_2 are disjoint open sets whose closures intersect in an $(n-1)$-dimensional manifold or *interface*. Additional examples will be given in Section 5.

2 Regular Equations

Let V and W be Hilbert spaces with V a dense subspace of W for which the injection is continuous. Thus, we identify $W' \leq V'$ by duality. Let $\mathcal{A} \in \mathcal{L}(V, V')$ and $\mathcal{C} \in \mathcal{L}(W, W')$ be given. Suppose $D(B) \leq V$ and $B : D(B) \to V'$ is linear. If $u_0 \in V$, $u_1 \in W$ and $f \in C((0,\infty), W')$ are given, we consider the problem of finding $u \in C([0,\infty), V) \cap C^1((0,\infty), V) \cap C^1([0,\infty), W) \cap C^2((0,\infty), W)$ such that $u(0) = u_0$, $u'(0) = u_1$, and

$$\mathcal{C}u''(t) + Bu'(t) + \mathcal{A}u(t) = f(t) \tag{2.1}$$

for all $t > 0$. Note that for any such solution of (2.1) we have $u'(t) \in D(B)$ and $Bu'(t) + \mathcal{A}u(t) \in W'$ for all $t > 0$.

We shall solve (2.1) by reducing it to a first order equation on a product space and then applying the results of Section V.2. The idea is to write (2.1) in the form

$$\begin{pmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{C} \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}' + \begin{pmatrix} 0 & -\mathcal{A} \\ \mathcal{A} & B \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} 0 \\ f(t) \end{pmatrix} \ .$$

Define $V_m = V \times W$, the product Hilbert space with scalar-product given by

$$([x_1, x_2], [y_1, y_2])_{V_m} = (x_1, y_1)_V + (x_2, y_2)_W \ , \qquad [x_1, x_2], [y_1, y_2] \in V \times W \ .$$

We have then $V'_m = V' \times W'$, and we define $\mathcal{M} \in \mathcal{L}(V_m, V'_m)$ by

$$\mathcal{M}([x_1, x_2]) = [\mathcal{A}x_1, \mathcal{C}x_2] \ , \qquad [x_1, x_2] \in V_m \ .$$

Define $D = \{[x_1, x_2] \in V \times D(B) : \mathcal{A}x_1 + Bx_2 \in W'\}$ and $L \in L(D, V'_m)$ by

$$L([x_1, x_2]) = [-\mathcal{A}x_2, \mathcal{A}x_1 + Bx_2] \ , \qquad [x_1, x_2] \in D \ .$$

If $u(\cdot)$ is a solution of (2.1), then the function defined by $w(t) = [u(t), u'(t)]$, $t \geq 0$, satisfies the following: $w \in C([0, \infty), V_m) \cap C^1((0, \infty), V_m)$, $w(0) = [u_0, u_1] \in V_m$, and

$$\mathcal{M}w'(t) + Lw(t) = [0, f(t)] \ , \qquad t > 0 \ . \tag{2.2}$$

This is precisely the situation of Section V.2, so we need only to find conditions on the data in (2.1) so that Theorems 2.1 or 2.2 of Chapter V are applicable. This leads to the following.

Theorem 2.1 *Let V and W be Hilbert spaces with V dense and continuously imbedded in W. Assume $\mathcal{A} \in \mathcal{L}(V, V')$ and $\mathcal{C} \in \mathcal{L}(W, W')$ are the Riesz maps of V and W, respectively, and let B be linear from the subspace $D(B)$ of V into V'. Assume that B is monotone and that $\mathcal{A} + B + \mathcal{C} : D(B) \to V'$ is surjective. Then for every $f \in C^1([0, \infty), W')$ and $u_0 \in V$, $u_1 \in D(B)$ with $\mathcal{A}u_0 + Bu_1 \in W'$, there exists a unique solution $u(t)$ of* (2.1) *(on $t \geq 0$) with $u(0) = u_0$ and $u'(0) = u_1$.*

Proof: Since $\mathcal{A}$ and $\mathcal{C}$ are Riesz maps of their corresponding spaces, we have

$$\begin{aligned} \mathcal{M}([x_1, x_2])([y_1, y_2]) &= \mathcal{A}x_1(y_1) + \mathcal{C}x_2(y_2) \\ &= (x_1, y_1)_V + (x_2, y_2)_W \\ &= ([x_1, x_2], [y_1, y_2])_{V_m} \ , \qquad [x_1, x_2], [y_1, y_2] \in V_m \ , \end{aligned}$$

so $\mathcal{M}$ is the Riesz map of V_m. Also we have for $[x_1, x_2] \in D$

$$L([x_1, x_2])([y_1, y_2]) = -\mathcal{A}x_2(y_1) + (\mathcal{A}x_1 + Bx_2)(y_2) \ , \qquad [y_1, y_2] \in V_m \ ,$$

hence, $L([x_1,x_2])([x_1,x_2]) = -\overline{\mathcal{A}x_1(x_2)} + \mathcal{A}x_1(x_2) + Bx_2(x_2)$ since $\mathcal{A}$ is symmetric. From this we obtain

$$\operatorname{Re} L([x_1,x_2])([x_1,x_2]) = \operatorname{Re} Bx_2(x_2) \ , \qquad [x_1,x_2] \in D \ ,$$

so B being monotone implies L is monotone. Finally, if $f_1 \in V'$ and $f_2 \in W'$, then we can find $x_2 \in D(B)$ such that $(\mathcal{A} + B + \mathcal{C})x_2 = f_2 - f_1$. Setting $x_1 = x_2 + \mathcal{A}^{-1}f_1 \in V$, we have a pair $[x_1,x_2] \in D$ for which $(\mathcal{M} + L)[x_1,x_2] = [f_1,f_2]$. (Note that $\mathcal{A}x_1 + Bx_2 = f_2 - \mathcal{C}x_2 \in W'$ as required.) Thus $\mathcal{M} + L$ is a surjection of D onto V_m'. Theorem 2.1 of Chapter V asserts the existence of a solution $w(t) = [u(t), v(t)]$ of (2.2). Since $\mathcal{A}$ is a norm-preserving isomorphism, $v(t) = u'(t)$ and the result follows.

A special case of Theorem 2.1 that occurs frequently in applications is that $D(B) = V$ and $B = \mathcal{B} \in \mathcal{L}(V,V')$. Then one needs only to verify that $\mathcal{B}$ is monotone, for then $\mathcal{A}+\mathcal{B}+\mathcal{C}$ is V-coercive, hence surjective. Furthermore, in this case we may define $\mathcal{L} \in \mathcal{L}(V_\ell, V_\ell')$ and $V_\ell = V \times V$ by

$$\mathcal{L}([x_1,x_2])([y_1,y_2]) = -\mathcal{A}x_2(y_1) + (\mathcal{A}x_1 + \mathcal{B}x_2)(y_2) \ , \qquad [x_1,x_2],[y_1,y_2] \in V_\ell \ .$$

Thus, Theorem 2.2 of Chapter V applies if we can show that $\mathcal{L}(\cdot)(\cdot)$ is V_ℓ-elliptic. Of course we need only to verify that $(\lambda\mathcal{M} + \mathcal{L})(\cdot)(\cdot)$ is V_ℓ-elliptic for some $\lambda > 0$ (Exercise V.2.3), and this leads us to the following.

Theorem 2.2 *Let $\mathcal{A}$ and $\mathcal{C}$ be the Riesz maps of the Hilbert spaces V and W, respectively, where V is dense and continuously imbedded in W. Let $\mathcal{B} \in \mathcal{L}(V,V')$ and assume $\mathcal{B} + \lambda\mathcal{C}$ is V-elliptic for some $\lambda > 0$. Then for every Hölder continuous $f : [0,\infty) \to W'$, $u_0 \in V$ and $u_1 \in W$, there is a unique solution $u(t)$ of* (2.1) *on $t > 0$ with $u(0) = u_0$ and $u'(0) = u_1$.*

Theorem 2.2 applies to evolution equations of second order which are parabolic, i.e., those which can be solved for more general data u_0, u_1 and $f(\cdot)$, and whose solutions are smooth for all $t > 0$. Such problems occur when energy is strongly dissipated; we give examples below. The situation in which energy is conserved is described in the following result. We leave its proof as an exercise, as it is a direct consequence of either Theorem 2.2 above or Section IV.5.

Theorem 2.3 *In addition to the hypotheses of Theorem 2.1, assume that $\operatorname{Re} Bx(x) = 0$ for all $x \in D(B)$ and that both $\mathcal{A} + B + \mathcal{C}$ and $\mathcal{A} - B + \mathcal{C}$ are* *surjections of* $D(B)$ *onto* V'. *Then for every* $f \in C^1(\mathbb{R}, W')$ *and* $u_0 \in V$, $u_1 \in D(B)$ *with* $\mathcal{A}u_0 + Bu_1 \in W'$, *there exists a unique solution of* (2.1) *on* $\mathbb{R}$ *with* $u(0) = u_0$ *and* $u'(0) = u_1$.

We shall describe how Theorems 2.1 and 2.3 apply to an *abstract wave equation*. Examples will be given afterward. Assume we are given Hilbert spaces $V \leq H$, and B, and a linear surjection $\gamma : V \to B$ with kernel V_0 such that γ factors into an isomorphism of V/V_0 onto B, the injection $V \hookrightarrow H$ is continuous, V_0 is dense in H, and H is identified with its dual H' by the Riesz map. We thereby obtain continuous injections $V_0 \hookrightarrow H \hookrightarrow V_0'$ and $V \hookrightarrow H \hookrightarrow V'$.

Let $a_1 : V \times V \to \mathbb{K}$ and $a_2 : B \times B \to \mathbb{K}$ be continuous symmetric sesquilinear forms and define $a : V \times V \to \mathbb{K}$ by

$$a(u,v) = a_1(u,v) + a_2(\gamma(u), \gamma(v)) \ , \qquad u, v \in V \ . \tag{2.3}$$

Assume $a(\cdot,\cdot)$ is V-elliptic. Then $a(\cdot,\cdot)$ is a scalar-product on V which gives an equivalent norm on V, so we hereafter consider V with this scalar-product, i.e., $(u,v)_V \equiv a(u,v)$ for $u, v \in V$. The form (2.3) will be used to prescribe an abstract boundary value problem as in Section III.3. Thus, we define $A : V \to V_0'$ by

$$Au(v) = a_1(u,v) \ , \qquad u \in V \ , \ v \in V_0$$

and $D_0 = \{u \in V : Au \in H\}$. Then Theorem III.2.3 gives the abstract boundary operator $\partial_1 \in L(D_0, B')$ for which

$$a_1(u,v) - (Au,v)_H = \partial_1 u(\gamma v) \ , \qquad u \in D_0 \ , \ v \in V \ .$$

We define $D = \{u \in V : \mathcal{A}u \in \mathcal{H}\}$, where $\mathcal{A}$ is the Riesz map of V given by

$$\mathcal{A}u(v) = a(u,v) \ , \qquad u, v \in V \ ,$$

and $\mathcal{A}_2 : B \to B'$ is given by

$$\mathcal{A}_2\varphi(\psi) = a_2(\varphi, \psi) \ , \qquad \varphi, \psi \in B \ .$$

Then, we recall from Corollary III.3.2 that $u \in D$ if and only if $u \in D_0$ and $\partial_1 u + \mathcal{A}_2(\gamma u) = 0$.

Let $(\cdot,\cdot)_W$ be a scalar-product on H whose corresponding norm is equivalent to that of $(\cdot,\cdot)_H$, and let W denote the Hilbert space consisting of H with the scalar-product $(\cdot,\cdot)_W$. Then the Riesz map $\mathcal{C}$ of W satisfies $\mathcal{C} \in \mathcal{L}(H)$ (and $\mathcal{C}^{-1} \in \mathcal{L}(H)$). Suppose we are also given an operator $\mathcal{B} \in \mathcal{L}(V, H)$ which is monotone (since $H \leq V'$).

Theorem 2.4 *Assume we are given the Hilbert spaces V, H, B, V_0, W and linear operators γ, ∂_1, $\mathcal{A}_2$, A, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ as above. Then for every $f \in C^1([0,\infty), H)$, $u_0 \in D$ and $u_1 \in V$, there is a unique solution $u(\cdot)$ of* (2.1), *and it satisfies*

$$\left.\begin{array}{ll} \mathcal{C}u''(t) + \mathcal{B}u'(t) + Au(t) = f(t) \ , & t \geq 0 \ , \\ u(t) \in V \ , \quad \partial_1 u(t) + \mathcal{A}_2\gamma(u(t)) = 0 \ , & t \geq 0 \ , \\ u(0) = u_0 \ , \quad u'(0) = u_1 \ , & \end{array}\right\} \tag{2.4}$$

Proof: Since $\mathcal{A} + \mathcal{B} + \mathcal{C} \in \mathcal{L}(V, V')$ is V-elliptic, it is surjective so Theorem 2.1 (with $B = \mathcal{B}$) asserts the existence of a unique solution. Also, since each of the terms $\mathcal{C}u''(t)$, $\mathcal{B}u'(t)$ and $f(t)$ of the equation (2.1) are in H, it follows that $\mathcal{A}u(t) \in H$ and, hence, $u(t) \in D$. This gives the middle line in (2.4).

In each of our examples below, the first line in (2.4) will imply an abstract wave equation, possibly with damping, and the second line will imply boundary conditions.

2.1

Let G be open in $\mathbb{R}^n$ and take $H = L^2(G)$. Let $\rho \in L^\infty(G)$ satisfy $\rho(x) \geq c > 0$ for $x \in G$, and define

$$(u, v)_W \equiv \int_G \rho(x) u(x) \overline{v(x)} \, dx \ , \qquad u, v \in H \ .$$

Then $\mathcal{C}$ is just multiplication by $\rho(\cdot)$.

Suppose further that ∂G is a C^1 manifold and Γ is a closed subset of ∂G. We define $V = \{v \in H^1(G) : \gamma_0(v)(s) = 0, \text{ a.e., } s \in \Gamma\}$, $\gamma = \gamma_0|_V$ and, hence, $V_0 = H_0^1(G)$ and B is the range of γ. Note that $B \hookrightarrow L^2(\partial G \sim \Gamma) \hookrightarrow B'$. We define

$$a_1(u, v) = \int_G \nabla u \cdot \nabla \bar{v} \, dx \ , \qquad u, v \in V \ ,$$

and it follows that $A = -\Delta_n$ and ∂_1 is the normal derivative

$$\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$$

on ∂G. Let $\alpha \in L^\infty(\partial G)$ satisfy $\alpha(s) \geq 0$, a.e. $s \in \partial G$, and define

$$a_2(\varphi, \psi) = \int_{\partial G \sim \Gamma} \alpha(s)\varphi(s)\overline{\psi(s)}\, ds\ , \qquad \varphi, \psi \in B\ .$$

Then $\mathcal{A}_2$ is multiplication by $\alpha(\cdot)$.

Assume that for each $t \in [0, T]$ we are given $F(\cdot, t) \in L^2(G)$, that $\partial_t F(x, t)$ is continuous in t for almost every $x \in G$, and $|\partial_t F(x, t)| \leq g(x)$ for some $g \in L^2(G)$. It follows that the map $t \mapsto F(\cdot, t) \equiv f(t)$ belongs to $C^1([0, T], L^2(G))$. Finally, let $U_0(\cdot) \in D$ (see below) and $U_1(\cdot) \in V$ be given. Then, if $u(\cdot)$ denotes the solution of (2.4) it follows from Theorem IV.7.1 that we can construct a function $U \in L^2(G \times [0, T])$ such that $U(\cdot, t) = u(t)$ in $L^2(G)$ for each $t \in [0, T]$ and this function satisfies the partial differential equation

$$\rho(x)\partial_t^2 U(x, t) - \Delta_n U(x, t) = F(x, t)\ , \qquad x \in G\ ,\ 0 \leq t \leq T \tag{2.5}$$

and the initial conditions

$$U(x, 0) = U_0(x)\ , \quad \partial_t U(x, 0) = U_1(x)\ , \quad \text{a.e. } x \in G\ .$$

Finally, from the inclusion $u(t) \in D$ we obtain the boundary conditions for $t \geq 0$

$$\left.\begin{aligned} &U(s, t) = 0\ , && \text{a.e. } s \in \Gamma \text{ and} \\ &\frac{\partial U(s, t)}{\partial \nu} + \alpha(s) U(s, t) = 0\ , && \text{a.e. } s \in \partial G \sim \Gamma\ . \end{aligned}\right\} \tag{2.6}$$

The first equation in (2.6) is the boundary condition of *first type*. The second is the boundary condition of *second type* where $\alpha(s) = 0$ and of *third type* where $\alpha(s) > 0$. (Note that U_0 necessarily satisfies the conditions of (2.6) with $t = 0$ and that U_1 satisfies the first condition in (2.6). If $F(\cdot, t)$ is given as above but for each $t \in [-T, T]$, then Theorem 2.3 (and Theorem III.7.5) give a solution of (2.5) on $G \times [-T, T]$.

2.2

In addition to all the data above, suppose we are given $R(\cdot) \in L^\infty(G)$ and a vector field $\mu(x) = (\mu_1(x), \ldots, \mu_n(x))$, $x \in G$, with each $\mu_j \in C^1(\bar{G})$. We define $\mathcal{B} \in \mathcal{L}(V, H)$ (where $V \leq H^1(G)$ and $H = L^2(G)$) by

$$\mathcal{B}u(v) = \int_G \left(R(x)u(x) + \frac{\partial u(x)}{\partial \mu} \right) \overline{v(x)}\, dx \tag{2.7}$$

the indicated directional derivative being given by

$$\frac{\partial u(x)}{\partial \mu} \equiv \sum_{j=1}^{n} \partial_j u(x) \mu_j(x) \ .$$

From the Divergence Theorem we obtain

$$2 \operatorname{Re} \int_G \frac{\partial u(x)}{\partial \mu} \overline{u(x)} \, dx + \int_G \left(\sum_{j=1}^{n} \partial_j \mu_j(x) \right) |u(x)|^2 \, dx = \int_{\partial G} (\mu \cdot \nu) |u(x)|^2 \, ds \ ,$$

where $\mu \cdot \nu = \sum_{j=1}^{n} \mu_j(s) \nu_j(s)$ is the indicated euclidean scalar-product. Thus, $\mathcal{B}$ is monotone if

$$-\left(\tfrac{1}{2}\right) \sum_{j=1}^{n} \partial_j \mu_j(x) + \operatorname{Re}\{R(x)\} \geq 0 \ , \qquad x \in G$$

$$\mu(s) \cdot \nu(s) \geq 0 \ , \qquad s \in \partial G \sim \Gamma \ .$$

The first equation represents friction or energy dissipation distributed throughout G and the second is friction distributed over ∂G. Note that these are determined by the divergence of μ and the normal component of μ, respectively. If $u(\cdot)$ is a solution of (2.4) and the corresponding $U(\cdot, \cdot)$ is obtained as before from Theorem IV.7.1, then $U(\cdot, \cdot)$ is a generalized solution of the initial-boundary value problem

$$\begin{cases} \rho(x) \partial_t^2 U(x,t) + R(x) \partial_t U(x,t) + \partial_t \dfrac{\partial U(x,t)}{\partial \mu} - \Delta_n U(x,t) = F(x,t) \ , \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in G \ , \ t \geq 0 \\ U(s,t) = 0 \ , \qquad \text{a.e. } \ s \in \Gamma \ , \\ \dfrac{\partial U(s,t)}{\partial \nu} + \alpha(s) U(s,t) = 0 \ , \qquad \text{a.e. } \ s \in \partial G \sim \Gamma \\ U(x,0) = U_0(x) \ , \quad \partial_t U(x,0) = U_1(x) \end{cases}$$

One could similarly solve problems with the fourth boundary condition, oblique derivatives, transition conditions on an interface, etc., as in Section III.4. We leave the details as exercises.

We now describe how Theorem 2.2 applies to an *abstract viscoelasticity equation*.

Theorem 2.5 *Assume we are given the Hilbert spaces V, H, B, V_0, W and linear operators γ, ∂_1, $\mathcal{A}_2$, A, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ as in Theorem 2.4. Then for every $f : [0, \infty) \to H$ which is Hölder continuous, $u_0 \in V$ and $u_1 \in H$, there is a unique solution $u(t)$ of* (2.1) *with $B = \mathcal{B} + \varepsilon\mathcal{A}$ and $\varepsilon > 0$. This solution satisfies*

$$\left.\begin{array}{ll} \mathcal{C}u''(t) + (\mathcal{B} + \varepsilon A)u'(t) + Au(t) = f(t) \ , & t > 0 \ , \\ u(t) \in V \ , & t \geq 0 \ , \\ \partial_1(\varepsilon u'(t) + u(t)) + \mathcal{A}_2\gamma(\varepsilon u'(t) + u(t)) = 0 \ , & t > 0 \ , \\ u(0) = u_0 \ , \quad u'(0) = u_1 \ . & \end{array}\right\} \tag{2.8}$$

Proof: This follows immediately from

$$\operatorname{Re} Bx(x) \geq \varepsilon\mathcal{A}x(x) = \varepsilon\|x\|_V^2 \ , \qquad x \in V \ ,$$

(since $\mathcal{B}$ is monotone) and the observation that $\varepsilon u'(t) + u(t) \in D$ for $t > 0$.

2.3

Let all spaces and operators be chosen just as in Section 2.1 above. Suppose $U_0 \in V$, $U_1 \in H$ and $f(t) = F(t, \cdot)$, $t \geq 0$, where $F(\cdot, \cdot)$ is given as in Theorem IV.7.3. Then we obtain a generalized solution of the initial-boundary value problem

$$\left.\begin{array}{l} \rho(x)\partial_t^2 U(x,t) - \varepsilon\partial_t\Delta_n U(x,t) - \Delta_n U(x,t) = F(x,t) \ , \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } x \in G \ , \ t > 0 \ , \\ U(s,t) = 0 \ , \qquad \text{a.e. } s \in \Gamma \ , \ t \geq 0 \ , \\ \dfrac{\partial}{\partial\nu}(\varepsilon\partial_t U(s,t) + U(s,t)) + \alpha(s)(\varepsilon\partial_t U(s,t) + U(s,t)) = 0 \ , \\ \qquad\qquad\qquad\qquad\qquad \text{a.e. } s \in \partial G \sim \Gamma \ , \ t > 0 \ , \\ U(x,0) = U_0(x) \ , \quad \partial_t U(x,0) = U_1(x) \ , \qquad x \in G \ . \end{array}\right\} \tag{2.9}$$

In certain applications the coefficient $\varepsilon > 0$ corresponds to *viscosity* in the model and it distinguishes the preceding parabolic problem from the corresponding hyperbolic problem in Section 2.1. Problems with viscosity result in very strong damping effects on solutions. Dissipation terms of lower order like (2.7) could easily be added to the system (2.9), and other types of boundary conditions could be obtained.

3 Sobolev Equations

We shall give sufficient conditions for a certain type of evolution equation to have either a weak solution or a strong solution, a situation similar to that for pseudoparabolic equations. The problems we consider here have the strongest operator as the coefficient of the term in the equation with the second order derivative.

Theorem 3.1 *Let V be a Hilbert space and $\mathcal{A},\mathcal{B},\mathcal{C} \in \mathcal{L}(V,V')$. Assume that the sesquilinear form corresponding to $\mathcal{C}$ is V-elliptic. Then for every $u_0, u_1 \in V$ and $f \in C(\mathbb{R},V)$ there is a unique $u \in C^2(\mathbb{R},V)$ such that*

$$\mathcal{C}u''(t) + \mathcal{B}u'(t) + \mathcal{A}u(t) = f(t)\ , \qquad t \in \mathbb{R}\ , \tag{3.1}$$

and $u(0) = u_0$, $u'(0) = u_1$.

Proof: The change of variable $v(t) \equiv e^{-\lambda t}u(t)$ gives an equivalent problem with $\mathcal{A}$ replaced by $\mathcal{A}+\lambda\mathcal{B}+\lambda^2\mathcal{C}$, and this last operator is V-coercive if λ is chosen sufficiently large. Hence, we may assume $\mathcal{A}$ is V-elliptic. If we define $\mathcal{M}$ and $\mathcal{L}$ as in Section 2.2, then $\mathcal{M}$ is $V \times V \equiv V_m$-elliptic, and Theorem V.3.1 then applies to give a solution of (2.2). The desired result then follows.

A solution $u \in C^2(\mathbb{R},V)$ of (3.1) is called a *weak solution*. If we are given a Hilbert space H in which V is continuously imbedded and dense, we define $D(C) = \{v \in V : \mathcal{C}v \in H\}$ and $C = \mathcal{C}|_{D(C)}$. The corresponding restrictions of $\mathcal{B}$ and $\mathcal{A}$ to H are denoted similarly. A (weak) solution u of (3.1) for which each term belongs to H at each $t \in \mathbb{R}$ is called a *strong solution*, and it satisfies

$$Cu''(t) + Bu'(t) + Au(t) = f(t)\ , \qquad t \in \mathbb{R}\ . \tag{3.2}$$

Theorem 3.2 *Let the Hilbert space V and operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be given as in Theorem 3.1. Let the Hilbert space H and corresponding operators A, B, C be defined as above, and assume $D(C) \subset D(A) \cap D(B)$. Then for every pair $u_0 \in D(A)$, $u_1 \in D(C)$, and $f \in C(\mathbb{R},H)$, there is a unique strong solution $u(\cdot)$ of (3.2) with $u(0) = u_0$, $u'(0) = u_1$.*

Proof: We define $M[x_1,x_2] = [Ax_1, Cx_2]$ on $D(A) \times D(C) = D(M)$ and $L[x_1,x_2] = [-Ax_2, Ax_1 + Bx_2]$ on $D(A) \times D(A) \cap D(B)$ and apply Theorem V.3.2.

Corollary 3.3 *In the situation of Theorem 3.2, the weak solution $u(\cdot)$ is strong if and only if, for some $t_0 \in \mathbb{R}$, $u(t_0) \in D(A)$ and $u'(t_0) \in D(C)$.*

We give an example which includes the classical Sobolev equation from fluid mechanics and an evolution equation of the type used to describe certain vibration problems. Let G be open in $\mathbb{R}^n$ and suppose that ∂G is a C^1 manifold and that Γ is a closed subset of ∂G. Let $V = \{v \in H^1(G) : \gamma v(s) = 0, \text{ a.e. } s \in \Gamma\}$ and

$$\mathcal{C}u(v) = (u, v)_{H^1(G)} , \qquad u, v \in V .$$

Suppose $a_j(\cdot) \in L^\infty(G)$ for $j = 1, 2, \ldots, n$, and define

$$\mathcal{A}u(v) = \sum_{j=1}^{n} \int_G a_j(x) \partial_j u(x) \partial_j \overline{v(x)}\, dx , \qquad u, v \in V .$$

Let the functions $t \mapsto F(\cdot, t) : \mathbb{R} \to L^2(G)$ and $t \mapsto g(\cdot, t) : \mathbb{R} \to L^2(\partial G)$ be continuous and define $f \in C(\mathbb{R}, V')$ by

$$f(t)(v) = \int_G F(x,t)\overline{v(x)}\, dx + \int_{\partial G} g(s,t) \gamma \overline{v(s)}\, ds , \qquad v \in V .$$

Then for each pair $U_0, U_1 \in V$, we obtain from Theorems 3.1 and IV.7.1 a unique generalized solution of the problem

$$\left.\begin{aligned}
&\partial_t^2 U(x,t) - \Delta_n \partial_t^2 U(x,t) - \sum_{j=1}^{n} \partial_j (a_j(x) \partial_j U(x,t)) = F(x,t) , \\
&\hspace{20em} x \in G , \ t > 0 , \\
&U(s,t) = 0 , \qquad s \in \Gamma , \\
&\partial_\nu \partial_t^2 U(s,t) + \sum_{j=1}^{n} a_j(s) \partial_j U(s,t) = g(s,t) , \qquad s \in \partial G \sim \Gamma , \\
&U(x,0) = U_0(x) , \quad \partial_t U(x,0) = U_1(x) .
\end{aligned}\right\} \tag{3.3}$$

In the special case of $a_j \equiv 0$, $1 \le j \le n-1$, and $a_n(x) \equiv 1$, the partial differential equation in (3.3) is *Sobolev's equation* which describes inertial waves in rotating fluids. Terms due to temperature gradients will give (3.3) with $a_j(x) \equiv a > 0$, $1 \le j \le n-1$, and $a_n(x) \equiv 1$. Finally, if $a_j(x) \equiv a > 0$, $1 \le j \le n$, then the partial differential equation in (3.3) is *Love's equation* for longitudinal vibrations with lateral inertia.

Suppose now that $g \equiv 0$ in the above, hence, $f \in C(\mathbb{R}, H)$, where $H = L^2(G)$. If we assume $\Gamma = \partial G$, hence, $V = H_0^1(G)$, then $D(C) = H_0^1(G) \cap H^2(G) \subset D(A)$, so Theorem 3.2 gives a smoother solution of (3.3) whenever $U_0, U_1 \in D(C)$. If instead we assume $a_j(x) \equiv a$, $1 \le j \le n$, then $D(C) = D(A)$, and Theorem 3.2 gives a smoother solution of (3.3) whenever $U_0, U_1 \in D(C)$.

Similar problems containing dissipation effects can easily be added, and we leave these to the exercises. In particular, there is motivation to consider problems like (3.3) with viscosity.

4 Degenerate Equations

We shall consider evolution equations of the form (2.1) wherein the leading operator $\mathcal{C}$ may not necessarily be the Riesz map of a Hilbert space. In particular, certain applications lead to (2.1) with $\mathcal{C}$ being symmetric and monotone. Our plan is to first solve a first order system like (2.2) by using one of Theorems V.4.1 or V.4.2. Then the first and second components will be solutions (of appropriate modifications) of (2.1). Also we shall obtain well-posed problems for a first order evolution equation in which the leading operator is not necessarily symmetric. (The results of Section V.4 do not apply to such a situation.)

4.1

Let $\mathcal{A}$ be the Riesz map of a Hilbert space V to its dual V'. Let $\mathcal{C} \in \mathcal{L}(V, V')$ and suppose its sesquilinear form is symmetric and non-negative on V. Then it follows (cf., Section V.4) that $x \mapsto \mathcal{C}x(x)^{1/2}$ is a seminorm on V; let W denote the corresponding seminorm space. Finally, suppose $D(B) \le V$ and $B \in L(D(B), V')$ are given. Now we define V_m to be the product $V \times W$ with the seminorm induced by the symmetric and non-negative sesquilinear form

$$m(x, y) = \mathcal{A}x_1(y_1) + \mathcal{C}x_2(y_2) \ , \qquad x, y \in V_m \equiv V \times W \ .$$

The identity $\mathcal{M}x(y) = m(x, y)$, $x, y \in V_m$, defines $\mathcal{M} \in \mathcal{L}(V_m, V_m')$. Finally we define $D \equiv \{[x_1, x_2] \in V \times D(B) : \mathcal{A}x_1 + Bx_2 \in W'\}$ and the linear map $L : D \to V_m'$ by

$$L[x_1, x_2] = [-\mathcal{A}x_2, \mathcal{A}x_1 + Bx_2] \ .$$

We shall apply Theorem V.4.1 to obtain the following result.

Theorem 4.1 *Let $\mathcal{A}$ be the Riesz map of the Hilbert space V and let W be the seminorm space obtained from a symmetric and monotone $\mathcal{C} \in \mathcal{L}(V, V')$. Let $D(B) \leq V$ and $B \in L(D(B), V')$ be monotone. Assume $B+\mathcal{C}$ is strictly monotone and $\mathcal{A}+B+\mathcal{C} : D(B) \to V'$ is a surjection. Let $f \in C^1([0,\infty), W')$ and $g \in C^1([0,\infty), V')$. If V_m and D are the spaces denoted above, then for every pair $[u_0, u_1] \in D$ there exists a unique function $w(\cdot) : [0,\infty) \to D$ such that $\mathcal{M}w(\cdot) \in C^1([0,\infty), V_m')$, $\mathcal{M}w(0) = \mathcal{M}[u_0, u_1]$, and*

$$(\mathcal{M}w)'(t) + Lw(t) = [-g(t), f(t)] \ , \qquad t \geq 0 \ . \tag{4.1}$$

Proof: We need to verify that the hypotheses of Theorem V.4.1 are valid in this situation. First note that $K(\mathcal{M}) \cap D = \{[0, x_2] : x_2 \in D(B),\ Bx_2 \in W',\ \mathcal{C}x_2 = 0\}$. But if $y \in D(B)$ with $By \in W'$, then there is a $K \geq 0$ such that

$$|By(x)| \leq K|\mathcal{C}x(x)|^{1/2} \ , \qquad x \in V \ ,$$

hence, $|By(y)| \leq K|\mathcal{C}y(y)|^{1/2} = 0$ if $\mathcal{C}y = 0$. Thus, we have shown that

$$\mathrm{Re}(B + \mathcal{C})x_2(x_2) = 0 \ , \qquad x = [0, x_2] \in K(\mathcal{M}) \cap D \ ,$$

so $B + \mathcal{C}$ being strictly monotone implies that $K(\mathcal{M}) \cap D = \{[0, 0]\}$. Finally, just as in the proof of Theorem 2.1, it follows from $\mathcal{A}+B+\mathcal{C}$ being surjective that $\mathcal{M} + L$ is surjective, so all the hypotheses of Theorem V.4.1 are true.

Let $w(\cdot)$ be the solution of (4.1) from Theorem 4.1 and set $w(t) = [u(t), v(t)]$ for each $t \geq 0$. If we set $g \equiv 0$ and eliminate $v(\cdot)$ from the system (4.1), then we obtain an equivalent second order evolution equation which $u(\cdot)$ satisfies and, thereby, the following result.

Corollary 4.2 *Let the spaces and operators be given as in Theorem 4.1. For every $f \in C^1([0,\infty), W')$ and every pair $u_0 \in V$, $u_1 \in D(B)$ with $\mathcal{A}u_0 + Bu_1 \in W'$ there exists a unique $u(\cdot) \in C^1([0,\infty), V)$ such that $\mathcal{C}u'(\cdot) \in C^1([0,\infty), W')$, $u(0) = u_0$, $\mathcal{C}u'(0) = \mathcal{C}u_1$, and for each $t \geq 0$, $u'(t) \in D(B)$, $\mathcal{A}u(t) + Bu'(t) \in W'$, and*

$$(\mathcal{C}u'(t))' + Bu'(t) + \mathcal{A}u(t) = f(t) \ . \tag{4.2}$$

Similarly, the function $v(\cdot)$ obtained from a solution of (4.1) satisfies a second order equation.

Corollary 4.3 *Let the spaces and operators be given as in Theorem 4.1. If $F \in C([0,\infty), W')$, $g \in C^1([0,\infty), V')$, $u_1 \in D(B)$ and $U_2 \in W'$, then there exists a unique $v(\cdot) : [0,\infty) \to D(B)$ such that $\mathcal{C}v(\cdot) \in C^1([0,\infty), W')$, $(\mathcal{C}v)' + Bv(\cdot) \in C^1([0,\infty), V')$, $\mathcal{C}v(0) = \mathcal{C}u_1$, $(\mathcal{C}v' + Bv)(0) = U_2 + Bu_1$, and for each $t \geq 0$,*

$$((\mathcal{C}v)'(t) + Bv(t))' + \mathcal{A}v(t) = F(t) + g(t) \ . \tag{4.3}$$

Proof: Given $F(\cdot)$ as above, define $f(\cdot) \in C^1([0,\infty), W')$ by $f(t) = \int_0^t F$. With u_1 and U_2 as above, there is a unique $u_0 \in V$ for which $\mathcal{A}u_0 = -Bu_1 - U_2$. Thus, $\mathcal{A}u_0 + Bu_1 \in W'$ so Theorem 4.1 gives a unique $w(\cdot)$ as indicated. Letting $w(t) \equiv [u(t), v(t)]$ for $t \geq 0$, we have immediately $v(t) \in D(B)$ for $t \geq 0$, $\mathcal{C}v \in C^1([0,\infty), W')$ and $\mathcal{C}v(0) = \mathcal{C}u_1$. The second line of (4.1) shows

$$(\mathcal{C}v)' + Bv = f - \mathcal{A}u \in C^1([0,\infty), V')$$

and the choice of u_0 above gives $(\mathcal{C}v)'(0) + Bv(0) = U + Bu_1$. Eliminating $u(\cdot)$ from (4.1) gives (4.3). This establishes the existence of $v(\cdot)$. The uniqueness result follows by defining $u(\cdot)$ by the second line of (4.1) and then noting that the function defined by $w(t) \equiv [u(t), v(t)]$ is a solution of (4.1).

Finally, we record the important special case of Corollary 4.3 that occurs when $\mathcal{C} = 0$. This leads to a well-posed problem for a first order equation whose leading operator is not necessarily symmetric.

Corollary 4.4 *Let the spaces V, $D(B)$ and operators B, $\mathcal{A}$ be given as in Theorem 4.1 but with $\mathcal{C} = 0$, hence, $W' = \{0\}$. Then for every $g \in C^1([0,\infty), V')$ and $u_1 \in D(B)$, there exists a unique $v : [0,\infty) \to D(B)$ such that $Bv(\cdot) \in C^1([0,\infty), V')$, $Bv(0) = Bu_1$, and for each $t \geq 0$,*

$$(Bv)'(t) + \mathcal{A}v(t) = g(t) \ . \tag{4.4}$$

4.2

Each of the preceding results has a parabolic analogue. We begin with the following.

Theorem 4.5 *Let $\mathcal{A}$ be the Riesz map of the Hilbert space V and let W be the seminorm space obtained from a symmetric and monotone $\mathcal{C} \in \mathcal{L}(V, V')$. Let $\mathcal{B} \in \mathcal{L}(V, V')$ be monotone and assume that $\mathcal{B} + \lambda\mathcal{C}$ is V-elliptic for some*

$\lambda > 0$. Then for every pair of Hölder continuous functions $f : [0,\infty) \to W'$, $g : [0,\infty) \to V'$ and each pair $u_0 \in V$, $U_1 \in W'$, there exists a unique function $w : [0,\infty) \to V_m$ such that $\mathcal{M}w(\cdot) \in C([0,\infty), V'_m) \cap C^1((0,\infty), V'_m)$, $\mathcal{M}w(0) = [\mathcal{A}u_0, U_1]$, and for all $t > 0$,

$$(\mathcal{M}w)'(t) + \mathcal{L}w(t) = [-g(t), f(t)] \ ,$$

where $\mathcal{L} \in \mathcal{L}(V \times V, V' \times V')$ is defined by $\mathcal{L}[x_1, x_2] = [-\mathcal{A}x_2, \mathcal{A}x_1 + \mathcal{B}x_2]$, and $\mathcal{M}$ is given as in Theorem 4.1.

Proof: By introducing a change-of-variable, if necessary, we may replace $\mathcal{L}$ by $\lambda\mathcal{M} + \mathcal{L}$. Since for $x \equiv [x_1, x_2] \in V \times V$ we have

$$\mathrm{Re}(\lambda\mathcal{M} + \mathcal{L})x(x) = \lambda\mathcal{A}x_1(x_1) + (\mathcal{B} + \lambda\mathcal{C})x_2(x_2) \ ,$$

we may assume $\mathcal{L}$ is $V \times V$-elliptic. The desired result follows from Theorem V.4.2.

Corollary 4.6 *Let the spaces and operators be given as in Theorem 4.5. For every Hölder continuous $f : [0,\infty) \to W'$, $u_0 \in V$ and $U_1 \in W'$, there exists a unique $u(\cdot) \in C([0,\infty), V) \cap C^1((0,\infty), V)$ such that $\mathcal{C}u'(\cdot) \in C((0,\infty), W') \cap C^1((0,\infty), W')$, $u(0) = u_0$, $\mathcal{C}u'(0) = U_1$, and*

$$(\mathcal{C}u'(t))' + \mathcal{B}u'(t) + \mathcal{A}u(t) = f(t) \ , \qquad t > 0 \ . \tag{4.5}$$

Corollary 4.7 *Let the spaces and operators be given as in Theorem 4.5. Suppose $F : (0,\infty) \to W'$ is continuous at all but a finite number of points and for some $p > 1$ we have $\int_0^T \|F(t)\|^p_{W'}\,dt < \infty$ for all $T > 0$. If $g : [0,\infty) \to V'$ is Hölder continuous, $u_1 \in V$ and $U_2 \in V'$, then there is a unique function $v(\cdot) : [0,\infty) \to V$ such that $\mathcal{C}v \in C([0,\infty), W') \cap C^1((0,\infty), W')$, $(\mathcal{C}v)' + \mathcal{B}v \in C([0,\infty), V')$ and is continuously differentiable at all but a finite number of points, $\mathcal{C}v(0) = \mathcal{C}u_1$, $(\mathcal{C}v' + \mathcal{B}v)(0) = U_2 + \mathcal{B}u_1$, and*

$$((\mathcal{C}v)'(t) + \mathcal{B}v(t))' + \mathcal{A}v(t) = F(t) + g(t) \tag{4.6}$$

at those points at which the derivative exists.

Proof: Almost everything follows as in Corollary 4.3. The only difference is that we need to note that with $F(\cdot)$ as given above, the function $f(t) = \int_0^t F$

satisfies

$$\|f(t) - f(\tau)\|_{W'} \le \int_\tau^t \|F\|_{W'} \le |t-\tau|^{1/q} \Big(\int_\tau^t \|F\|_{W'}^p \Big)^{1/p}$$
$$\le |t-\tau|^{1/q} \Big(\int_0^T \|F\|_{W'}^p \Big)^{1/p} , \qquad 0 \le \tau \le t \le T ,$$

where $1/q = 1 - 1/p \ge 0$. Hence, f is Hölder continuous.

5 Examples

We shall illustrate some applications of our preceding results by various examples of initial-boundary value problems. In each such example below, the operator $\mathcal{A}$ will correspond to one of the elliptic boundary value problems described in Section III.4, and we refer to that section for the computations as well as occasional notations. Our emphasis here will be on the *types* of operators that can be chosen for the remaining coefficients in either of (4.2) or (4.3).

We begin by constructing the operator $\mathcal{A}$ from the abstract boundary value problem of Section III.3. Let V, H and B be Hilbert spaces and $\gamma : V \to B$ a linear surjection with kernel V_0, and assume γ factors into a norm-preserving isomorphism of V/V_0 onto B. Assume the injection $V \hookrightarrow H$ is continuous, V_0 is dense in H, and H is identified with H'. Then we obtain the continuous injections $V_0 \hookrightarrow H \hookrightarrow V_0'$ and $V \hookrightarrow H \hookrightarrow V'$ and

$$(f, v)_H = f(v) , \qquad f \in H , \ v \in V .$$

Let $a_1 : V \times V \to \mathbb{K}$ and $a_2 : B \times B \to \mathbb{K}$ be continuous, sesquilinear and symmetric forms and define

$$a(u, v) \equiv a_1(u, v) + a_2(\gamma u, \gamma v) , \qquad u, v \in V . \tag{5.1}$$

We shall assume $a(\cdot,\cdot)$ is V-elliptic; thus, $a(\cdot,\cdot)$ is a scalar-product on V whose norm is equivalent to the original one on V. Hereafter, we shall take $a(\cdot,\cdot)$ as the scalar-product on V; the corresponding Riesz map $\mathcal{A} \in \mathcal{L}(V, V')$ is given by

$$\mathcal{A}u(v) = a(u, v) , \qquad u, v \in V .$$

Similarly, we define $A \in \mathcal{L}(V, V_0')$ by

$$Au(v) = a_1(u, v) , \qquad u \in V , \ v \in V_0 , \tag{5.2}$$

Let $D_0 \equiv \{u \in V : Au \in H\}$, and denote by $\partial \in L(D_0, B')$ the abstract Green's operator constructed in Theorem III.2.3 and characterized by the identity

$$a_1(u,v) - (Au,v)_H = \partial u(\gamma(v)) \ , \qquad u \in D_0 \ , \ v \in V \ . \tag{5.3}$$

Finally, we denote by $\mathcal{A}_2 \in \mathcal{L}(B, B')$ the operator given by

$$\mathcal{A}_2\varphi(\psi) = a_2(\varphi, \psi) \ , \qquad \varphi, \psi \in B \ .$$

It follows from (5.1), (5.2) and (5.3) that

$$\mathcal{A}u(v) - (Au,v)_H = (\partial u + \mathcal{A}_2(\gamma u))(\gamma v) \ , \qquad u \in D_0 \ , \ v \in V \ , \tag{5.4}$$

and this identity will be used to characterize the weak or variational boundary conditions below.

Let $c : H \times H \to \mathbb{K}$ be continuous, non-negative, sesquilinear and symmetric; define the monotone $\mathcal{C} \in \mathcal{L}(H)$ by

$$\mathcal{C}u(v) = c(u,v) \ , \qquad u, v \in H \ ,$$

where $\mathcal{C}u \in H$ follows from $H' = H$. Note that the inclusion $W' \subset H$ follows from the continuity of the injection $H \hookrightarrow W$, where W is the space H with seminorm induced by $c(\cdot,\cdot)$. Finally let $\mathcal{B} \in \mathcal{L}(V, H)$ be a given monotone operator

$$\operatorname{Re} \mathcal{B}u(v) \geq 0 \ , \qquad u \in V \ , \ v \in H \ ,$$

and assume $\mathcal{C} + \mathcal{B}$ is strictly monotone:

$$(\mathcal{C} + \mathcal{B})u(u) = 0 \ \text{ only if } \ u = 0 \ .$$

Theorem 5.1 *Let the Hilbert spaces and operators be given as above. For every $f \in C^1([0,\infty), W')$ and every pair $u_0, u_1 \in V$ with $\mathcal{A}u_0 + \mathcal{B}u_1 \in W'$, there exists a unique $u \in C^1([0,\infty), V)$ such that $\mathcal{C}u' \in C^1([0,\infty), W')$, $u(0) = u_0$, $\mathcal{C}u'(0) = \mathcal{C}u_1$, and for each $t \geq 0$,*

$$(\mathcal{C}u'(t))' + \mathcal{B}u'(t) + Au(t) = f(t) \ , \tag{5.5}$$

$$u(t) \in D_0 \subset V \ , \quad \partial u(t) + \mathcal{A}_2\gamma(u(t)) = 0 \ . \tag{5.6}$$

Proof: The existence and uniqueness of $u(\cdot)$ follows from Corollary 4.2. With $\mathcal{C}$ and $\mathcal{B}$ as above (4.2) shows that $\mathcal{A}u(t) \in H$ for all $t \geq 0$, so (5.6) follows from Corollary III.3.2. (Cf. (5.4).) To be sure, the pair of equations (5.5), (5.6), is equivalent to (4.2).

We illustrate Theorem 5.1 in the examples following in Sections 5.1 and 5.2.

5.1

Let G be open in $\mathbb{R}^n$, $H = L^2(G)$, $\Gamma \subset \partial G$ and $V = \{v \in H^1(G) : \gamma_0(v)(s) = 0,\ \text{a.e. } s \in \Gamma\}$. Let $p \in L^\infty(G)$ with $p(x) \geq 0$, $x \in G$, and define

$$c(u,v) = \int_G p(x)u(x)\overline{v(x)}\,dx\ , \qquad u,v \in H\ . \tag{5.7}$$

Then $\mathcal{C}$ is multiplication by p and $W' = \{p^{1/2}v : v \in L^2(G)\}$. Let $R \in L^\infty(G)$ and the real vector field $\mu(x) = (\mu_1(x), \ldots, \mu_n(x))$ be given with each $\mu_j \in C^1(\bar{G})$; assume

$$-\left(\tfrac{1}{2}\right)\sum_{j=1}^n \partial_j\mu_j(x) + \mathrm{Re}\{R(x)\} \geq 0\ , \qquad x \in G\ ,$$

$$\left(\tfrac{1}{2}\right)\mu(s)\cdot\nu(s) \geq 0\ , \qquad s \in \partial G \sim \Gamma\ .$$

Then $\mathcal{B} \in \mathcal{L}(V,H)$ given by (2.7) is monotone. Furthermore, we shall assume

$$p(x) - \left(\tfrac{1}{2}\right)\sum_{j=1}^n \partial_j\mu_j(x) + \mathrm{Re}\{R(x)\} > 0\ , \qquad x \in G\ ,$$

and this implies $\mathcal{C} + \mathcal{B}$ is strictly-monotone.

Let $a_0, a_{ij} \in L^\infty(G)$, $1 \leq i,j \leq n$, and assume $a_0(x) \geq 0$, $a_{ij}(x) = \overline{a_{ji}(x)}$, $x \in G$, and that

$$a(u,v) \equiv \int_G \left\{ \sum_{i,j=1}^n a_{ij}(x)\partial_i u(x)\partial_j\overline{v(x)} + a_0(x)u(x)\overline{v(x)} \right\} dx \tag{5.8}$$

is V-coercive (cf. Section III.5). Then (5.8) is a scalar product on V whose norm is equivalent to that of $H^1(G)$ on V.

Let $F(\cdot,t) \in L^2(G)$ be given for each $t \geq 0$ such that $t \mapsto F(\cdot,t)$ belongs to $C^1([0,\infty), L^2(G))$ (cf. Section 2.1). Then $f(t) \equiv p^{1/2}F(\cdot,t)$ defines $f \in C^1([0,\infty), W')$. Finally, let $U_0, U_1 \in V$ satisfy $\mathcal{A}U_0 + \mathcal{B}U_1 \in W'$. (This can be translated into an elliptic boundary value problem.) Using Theorem IV.7.1, we can obtain a (measurable) function $U(\cdot,\cdot)$ on $G \times [0,\infty)$ which is a solution of the initial-boundary value problem

$$\partial_t(p(x)\partial_t U(x,t)) + R(x)\partial_t U(x,t) + \frac{\partial}{\partial\mu}(\partial_t U(x,t)) \tag{5.9}$$

$$-\sum_{j=1}^{n} \partial_j a_{ij}(x)\partial_i U(x,t) + a_0(x)U(x,t)$$

$$= p^{1/2}(x)F(x,t)\ , \qquad x \in G\ ,\ t \geq 0\ ;$$

$$\left.\begin{aligned} &U(s,t) = 0\ , && s \in \Gamma\ , \\ &\frac{\partial U(s,t)}{\partial \nu_A} = 0\ , && s \in \partial G \sim \Gamma\ ; \end{aligned}\right\} \tag{5.10}$$

$$\left.\begin{aligned} &U(x,0) = U_0(x)\ , \\ &p(x)\partial_t U(x,0) = p(x)U_1(x)\ , \qquad x \in G\ . \end{aligned}\right\} \tag{5.11}$$

We refer to Section III.4.1 for notation and computations involving the operators associated with the form (5.8).

The partial differential equation (5.9) is of mixed hyperbolic-parabolic type. Note that the initial conditions (5.11) imposed on the solution at $x \in G$ depend on whether $p(x) > 0$ or $p(x) = 0$. Also, the equation (5.9) is satisfied at $t = 0$, thereby imposing a compatibility condition on the initial data U_0, U_1. Finally, we observe that (5.10) contains the boundary condition of *first type* along Γ and the boundary condition of *second type* on $\partial G \sim \Gamma$.

5.2

Let H and $\mathcal{C}$ be given as in Section 5.1; let $V = H^1(G)$ and define $\mathcal{B}$ by (2.7) with $\mu \equiv 0$ and assume

$$\mathrm{Re}\{R(x)\} \geq 0\ , \qquad p(x) \geq 0\ ,$$
$$p(x) + \mathrm{Re}\{R(x)\} > 0\ , \qquad x \in G\ ,$$

as before. Define

$$\begin{aligned} a_1(u,v) &= \textstyle\int_G \nabla u \cdot \nabla \bar{v} && u, v \in V\ , \\ a_2(\varphi,\psi) &= \textstyle\int_{\partial G} \alpha(s)\varphi(s)\overline{\psi(x)}\, ds\ , && \varphi, \psi \in L^2(\partial G) \end{aligned}$$

where $\alpha \in L^\infty(\partial G)$, $\alpha(s) \geq 0$, a.e. $s \in \partial G$. Then $\mathcal{A}_2$ is multiplication by α. We assume that $a(\cdot,\cdot)$ given by (5.1) is V-coercive (cf. Corollary III.5.5). With $F(\cdot,\cdot)$, U_0, and U_1 as above, we obtain a unique generalized solution of the problem

$$\partial_t(p(x)\partial_t U(x,t)) + R(x)\partial_t U(x,t) - \Delta_n U(x,t) \tag{5.12}$$

$$
\begin{aligned}
&= p^{1/2}(x)F(x,t) \ , \qquad && x \in G \ , \ t \geq 0 \ , \\
\frac{\partial U(s,t)}{\partial \nu} &+ \alpha(s)U(s,t) = 0 \ , && s \in \partial G \ , \ t \geq 0 \ , \qquad (5.13)
\end{aligned}
$$

and (5.11). We note that at those $x \in G$ where $p(x) > 0$, (5.12) is a (hyperbolic) wave equation and (5.11) specifies initially U and $\partial_t U$, whereas at those $x \in G$ where $p(x) = 0$, (5.12) is a homogeneous (parabolic) diffusion equation and only U is specified initially. The condition (5.13) is the boundary condition of *third type*.

If we choose $V = \{v \in H^1(G) : \gamma_0(v) \text{ is constant}\}$ as in Section III.4.2 and prescribe everything else as above, then we obtain a solution of (5.12), (5.11) and the boundary condition of *fourth type*

$$
\left.
\begin{aligned}
&U(s,t) = h(t) \ , \qquad s \in \partial G \ , \\
&\int_{\partial G} \frac{\partial U(s,t)}{\partial \nu} \, ds + \int_{\partial G} \alpha(s) \, ds \cdot h(t) = 0 \ .
\end{aligned}
\right\} \qquad (5.14)
$$

Note that $h(\cdot)$ is an unknown in the problem. Boundary value problems with *periodic boundary conditions* can be put in the form of (5.14).

5.3

Let $H = L^2(G)$, $V = H^1(G)$, and define $\mathcal{A}$ as in Section 5.2. Set $\mathcal{B} \equiv 0$ and define

$$
c(u,v) = \int_G p(x)u(x)\overline{v(x)} \, dx + \int_{\partial G} \sigma(s)u(s)\overline{v(s)} \, ds \ , \qquad u,v \in V
$$

when $p \in L^\infty(G)$ and $\sigma \in L^\infty(\partial G)$ satisfy $p(x) > 0$, $x \in G$, and $\sigma(s) \geq 0$, $s \in \partial G$. Let $t \mapsto F(\cdot,t)$ be given in $C^1([0,\infty), L^2(G))$ and $t \mapsto g(\cdot,t)$ be given in $C^1([0,\infty), L^2(\partial G))$; then define $f \in C^1([0,\infty), W')$ by

$$
f(t)(v) = \int_G p^{1/2}(x)F(x,t)\overline{v(x)} \, dx + \int_{\partial G} \sigma^{1/2}(s)g(s,t)\overline{v(s)} \, ds \ , \quad v \in V \ , \ t \geq 0 \ .
$$

Let $U_0, U_1 \in V$ with $\mathcal{A}U_0 \in W'$. (This last inclusion is equivalent to requiring $\Delta_n U_0 = p^{1/2}H$ for some $H \in L^2(G)$ and $\partial_\nu U_0 + \alpha U_0 = \sigma^{1/2}h$ for some $h \in L^2(\partial G)$.) Then Corollary 4.2 applies to give a unique solution u of (4.2) with initial conditions. From this we obtain a solution U of the problem

$$\begin{cases} \partial_t(p(x)\partial_t U(x,t)) - \Delta_n U(x,t) = p^{1/2}(x)F(x,t) \ , \\ \qquad\qquad\qquad\qquad\qquad\qquad x \in G \ , \ t \geq 0 \ , \\ \partial_t(\sigma(s)\partial_t U(s,t)) + \partial_\nu U(s,t) + \alpha(s)U(s,t) = \sigma^{1/2}(s)g(s,t) \ , \\ \qquad\qquad\qquad\qquad\qquad\qquad s \in \partial G \ , \ t \geq 0 \ , \\ U(x,0) = U_0(x) \ , \quad \partial_t U(x,0) = U_1(x) \ . \end{cases}$$

The boundary condition is obtained *formally* since we do not know $\Delta_n U(\cdot,t) \in L^2(G)$ for all $t > 0$; hence, (5.3) is not directly applicable. Such boundary conditions arise in models of vibrating membranes (or strings) with boundaries (or ends) loaded with a mass distribution, thereby introducing an inertia term. Such problems could also contain mass distributions (or point loads) on internal regions. Similarly, internal or boundary damping can be included by appropriate choices of $\mathcal{B}$, and we illustrate this in the following example.

5.4

Let H, V, $\mathcal{A}$ and $\mathcal{C}$ be given as in Section 5.3. Assume $R \in L^\infty(G)$, $r \in L^\infty(\partial G)$ and that $\mathrm{Re}\{R(x)\} \geq 0$, $x \in G$, $\mathrm{Re}\{r(s)\} \geq 0$, $s \in \partial G$. We define $\mathcal{B} \in \mathcal{L}(V, V')$ by

$$\mathcal{B}u(v) = \int_G R(x)u(x)\overline{v(x)}\,dx + \int_{\partial G} r(s)u(s)\overline{v(s)}\,ds \ , \qquad u, v \in V \ .$$

We need only to assume $p(x) + \mathrm{Re}\{R(x)\} > 0$ for $x \in G$; then Corollary 4.3 is applicable. Let $t \mapsto F_1(\cdot,t)$ in $C([0,\infty), L^2(G))$, $t \mapsto G_1(\cdot,t)$ in $C^1([0,\infty), L^2(\partial G))$, and $t \mapsto G_2(t)$ in $C^1([0,\infty), L^2(G))$ be given. We then define $F \in C([0,\infty), W')$ and $g \in C^1([0,\infty), V')$ by

$$\begin{aligned} F(t) &= p^{1/2}F_1(\cdot,t) \ , \\ g(t)(v) &= \int_{\partial G} \sigma^{1/2}(s)G_1(s,t)\overline{v(s)}\,ds + \int_G G_2(x,t)\overline{v(x)}\,dx \ , \qquad v \in V \ . \end{aligned}$$

If $U_1 \in V$ and $V_1 \in L^2(G)$, and $V_2 \in L^2(\partial G)$, then $U_2 \in W'$ is defined by

$$U_2(v) = \int_G p^{1/2}(x)V_1(x)\overline{v(x)}\,dx + \int_{\partial G} \sigma^{1/2}(s)V_2(s)\overline{v(s)}\,ds \ , \qquad v \in V \ ,$$

and Corollary 4.3 gives a generalized solution of the following problem:

$$\begin{cases} \partial_t^2(p(x)U(x,t)) + \partial_t(R(x)U(x,t)) - \Delta_n U(x,t) \\ \qquad\qquad = p^{1/2}(x)F_1(x,t) + G_2(x,t) \ , \qquad x \in G \ , \\ \partial_t^2(\sigma(s)U(s,t)) + \partial_t(r(s)U(s,t)) + \partial_\nu U(s,t) + (s)U(s,t) \\ \qquad\qquad = \sigma^{1/2}(s)G_1(s,t) \ , \qquad s \in \partial G \ , \ t > 0 \ , \\ p(x)U(x,0) = p(x)U_1(x) \ , \\ \sigma(s)U(s,0) = \sigma(s)U_1(s) \ , \qquad s \in \partial G \\ \partial_t(p(x)U(x,0)) + R(x)U(x,0) = p^{1/2}(x)V_1(x) \ , \\ \partial_t(\sigma(s)U(s,0)) + r(s)U(s,0) = \sigma^{1/2}(s)V_2(s) \ . \end{cases}$$

The right side of the partial differential equation could contain singularities in x as well. When $\mathrm{Re}\{R(x)\} > 0$ in G, the preceding problem with $p \equiv 0$ and $\sigma \equiv 0$ is solved by Corollary 4.4.

Similarly one can obtain generalized solutions to boundary value problems containing partial differential equations of the type (3.3); that is, equations of the form (5.9) plus the fourth-order term $-\partial_t(\Delta_n \partial_t U(x,t))$. Finally, we record an abstract parabolic boundary value problem which is solved by using Corollary 4.6. Such problems arise in classical models of linear viscoelasticity (cf. (2.9)).

Theorem 5.2 *Let the Hilbert spaces and operators be given as in Theorem 5.1, except we do not assume $\mathcal{B} + C$ is strictly monotone. If $\varepsilon > 0$, $f : [0,\infty) \to W'$ is Hölder continuous, $u_0 \in V$ and $U_1 \in W'$, there exists a unique $u \in C([0,\infty), V) \cap C^1([0,\infty), V)$ such that $\mathcal{C}u' \in C([0,\infty), W') \cap C^1((0,\infty), W')$, $u(0) = u_0$, $\mathcal{C}u'(0) = U_1$, and (5.5), (5.6) hold for each $t > 0$.*

Exercises

1.1. Use the separation-of-variables technique to obtain a series representation for the solution u of (1.1) with $u(0,t) = u(\pi,t) = 0$, $u(x,0) = u_0(x)$ and $\partial_t u(x,0) = u_1(x)$.

1.2. Repeat the above for the viscoelasticity equation

$$\partial_t^2 u - \varepsilon \partial_t \Delta_n u - \Delta_n u = F(x,t) \ , \qquad \varepsilon > 0 \ .$$

1.3. Compare the convergence rates of the two series solutions obtained above.

2.1. Explain the identification $V'_m = V' \times W'$ in Section 2.1.

2.2. Use Theorem 2.1 to prove Theorem 2.3.

2.3. Use the techniques of Section 2 to deduce Theorem 2.3 from IV.5.

2.4. Verify that the function f in Section 2.1 belongs to $C^1([0,T], L^2(G))$.

2.5. Use Theorem 2.1 to construct a solution of (2.5) satisfying the fourth boundary condition. Repeat for each of the examples in Section III.4.

2.6. Add the term $\int_{\partial G} r(s)u(s)\overline{v(s)}\,ds$ to (2.7) and find the initial-boundary value problem that results.

2.7. Show that Theorem 2.1 applies to appropriate problems for the equation

$$\partial_t^2 u(x,t) + \partial_x^3 \partial_t u(x,t) - \partial_x^2 u(x,t) = F(x,t) \ .$$

2.8. Find some well-posed problems for the equation

$$\partial_t^2 u(x,t) + \partial_x^4 u(x,t) = F(x,t) \ .$$

3.1. Complete the proofs of Theorem 3.2 and Corollary 3.3.

3.2. Verify that (3.3) is the characterization of (3.1) with the given data.

4.1. Use Corollary 4.4 to solve the problem

$$\begin{aligned} &\partial_t \partial_x u(x,t) - \partial_x^2 u(x,t) = F(x,t) \\ &u(0,t) = cu(1,t) \\ &u(x,0) = u_0(x) \end{aligned}$$

for $|c| \le 1$, $c \neq 1$.

4.2. For each of the Corollaries of Section 4, give an example which illustrates a problem solved by that Corollary only.

5.1. In the proof of Theorem 5.1, verify that (4.2) is equivalent to the pair (5.5), (5.6).

5.2. In Section 5.1, show $\mathcal{C}+\mathcal{B}$ is strictly monotone, give sufficient conditions for (5.8) to be V-elliptic, and characterize the condition $\mathcal{A}U_0 + \mathcal{B}U_1 \in W'$ as requiring that U_0 satisfy an elliptic boundary value problem (cf. Section 5.3).

5.3. In Section 5.2, give sufficient conditions for $a(\cdot,\cdot)$ to be V-elliptic.

5.4. Show the following problem with periodic boundary conditions is well-posed: $\partial_t^2 u - \partial_x^2 u = F(x,t)$, $u(x,0) = u_0(x)$, $\partial_t u(x,0) = u_1(x,0)$, $u(0,t) = u(1,t)$, $\partial_x u(0,t) = \partial_x u(1,t)$. Generalize this to higher dimensions.

5.5. A vibrating string loaded with a point mass m at $x = \frac{1}{2}$ leads to the following problem: $\partial_t^2 u = \partial_x^2 u$, $u(0,t) = u(1,t) = 0$, $u(x,0) = u_0(x)$, $\partial_t u(x,0) = u_1(x)$, $u((\frac{1}{2})^-,t) = u((\frac{1}{2})^+,t)$, $m\partial_t^2 u(\frac{1}{2},t) = \partial_x u((\frac{1}{2})^+,t) - \partial_x u((\frac{1}{2})^-,t)$. Use the methods of Section 5.3 to show this problem is well-posed.

Chapter VII

Optimization and Approximation Topics

1 Dirichlet's Principle

When we considered elliptic boundary value problems in Chapter III we found it useful to pose them in a weak form. For example, the Dirichlet problem

$$\left.\begin{aligned} -\Delta_n u(x) &= F(x)\ , \qquad & x \in G\ ,\\ u(s) &= 0\ , & s \in \partial G \end{aligned}\right\} \tag{1.1}$$

on a bounded open set G in $\mathbb{R}^n$ was posed (and solved) in the form

$$u \in H_0^1(G)\ ; \quad \int_G \nabla u \cdot \nabla v \, dx = \int_G F(x) v(x) \, dx\ , \qquad v \in H_0^1(G)\ . \tag{1.2}$$

In the process of formulating certain problems of mathematical physics as boundary value problems of the type (1.1), integrals of the form appearing in (1.2) arise naturally. Specifically, in describing the displacement $u(x)$ at a point $x \in G$ of a stretched string ($n = 1$) or membrane ($n = 2$) resulting from a unit tension and distributed external force $F(x)$, we find the *potential energy* is given by

$$E(u) = \left(\tfrac{1}{2}\right) \int_G |\nabla u(x)|^2 \, dx - \int_G F(x) u(x) \, dx\ . \tag{1.3}$$

Dirichlet's principle is the statement that the solution u of (1.2) is that function in $H_0^1(G)$ at which the functional $E(\cdot)$ attains its minimum. That is, u is the solution of

$$u \in H_0^1(G) : E(u) \le E(v) \ , \qquad v \in H_0^1(G) \ . \tag{1.4}$$

To prove that (1.4) characterizes u, we need only to note that for each $v \in H_0^1(G)$

$$E(u+v) - E(u) = \int_G (\nabla u \cdot \nabla v - Fv)\, dx + \left(\tfrac{1}{2}\right) \int_G |\nabla v|^2\, dx$$

and the first term vanishes because of (1.2). Thus $E(u+v) \ge E(u)$ and equality holds only if $v \equiv 0$.

The preceding remarks suggest an alternate proof of the existence of a solution of (1.2), hence, of (1.1). Namely, we seek the element u of $H_0^1(G)$ at which the energy function $E(\cdot)$ attains its minimum, then show that u is the solution of (1.2). This program is carried out in Section 2 where we minimize functions more general than (1.3) over closed convex subsets of Hilbert space. These more general functions permit us to solve some nonlinear elliptic boundary value problems.

By considering convex sets instead of subspaces we obtain some elementary results on unilateral boundary value problems. These arise in applications where the solution is subjected to a one-sided constraint, e.g., $u(x) \ge 0$, and their solutions are characterized by variational inequalities. These topics are presented in Section 3, and in Section 4 we give a brief discussion of some optimal control problems for elliptic boundary value problems.

Finally, Dirichlet's principle provides a means of numerically approximating the solution of (1.2). We pick a convenient finite-dimensional subspace of $H_0^1(G)$ and minimize $E(\cdot)$ over this subspace. This is the Rayleigh-Ritz method and leads to an approximate algebraic problem for (1.2). This method is described in Section 5, and in Section 6 we shall obtain related approximation procedures for evolution equations of first or second order.

2 Minimization of Convex Functions

Suppose F is a real-valued function defined on a closed interval K (possibly infinite). If F is continuous and if either K is bounded or $F(x) \to +\infty$ as $|x| \to +\infty$, then F attains its minimum value at some point of K. This result will be extended to certain real-valued functions on Hilbert space and the notions developed will be extremely useful in the remainder of this chapter. An essential point is to characterize the minimum by the derivative of F. Throughout this section V is a real separable Hilbert space, K is a non-empty subset of V and $F : K \to \mathbb{R}$ is a function.

2.1

We recall from Section I.6 that the space V is weakly (sequentially) compact. It is worthwhile to consider subsets of V which inherit this property. Thus, K is called *weakly (sequentially) closed* if the limit of every weakly convergent sequence from K is contained in K. Since convergence (in norm) implies weak convergence, a weakly closed set is necessarily closed.

Lemma 2.1 *If K is closed and convex (cf. Section I.4.2), then it is weakly closed.*

Proof: Let x be a vector not in K. From Theorem I.4.3 there is an $x_0 \in K$ which is closest to x. By translation, if necessary, we may suppose $(x_0 + x)/2 = \theta$, i.e., $x = -x_0$. Clearly $(x, x_0) < 0$ so we need to show that $(z, x_0) \geq 0$ for all $z \in K$; from this the desired result follows easily. Since K is convex, the function $\varphi : [0,1] \to \mathbb{R}$ given by

$$\varphi(t) = \|(1-t)x_0 + tz - x\|_V^2 \ , \qquad 0 \leq t \leq 1 \ ,$$

has its minimum at $t = 0$. Hence, the right-derivative $\varphi^+(0)$ is non-negative, i.e.,

$$(x_0 - x, z - x_0) \geq 0 \ .$$

Since $x = -x_0$, this gives $(x_0, z) \geq \|x_0\|_V^2 > 0$.

The preceding result and Theorem I.6.2 show that each closed, convex and bounded subset of V is weakly sequentially compact. We shall need to consider situations in which K is not bounded (e.g., $K = V$); the following is then appropriate.

Definition. The function F has the *growth property* at $x \in K$ if, for some $R > 0$, $y \in K$ and $\|y - x\| \geq R$ implies $F(y) > F(x)$.

The continuity requirement that is adequate for our purposes is the following.

Definition. The function $F : K \to \mathbb{R}$ is *weakly lower-semi-continuous* at $x \in K$ if for every sequence $\{x_n\}$ in K which weakly converges to $x \in K$

we have $F(x) \leq \liminf F(x_n)$. [Recall that for any sequence $\{a_n\}$ in $\mathbb{R}$, $\liminf(a_n) \equiv \sup_{k\geq 0}(\inf_{n\geq k}(a_n))$.]

Theorem 2.2 *Let K be closed and convex and $F : K \to \mathbb{R}$ be weakly lower-semi-continuous at every point of K. If* (a) *K is bounded or if* (b) *F has the growth property at some point in K, then there exists an $x_0 \in K$ such that $F(x_0) \leq F(x)$ for all $x \in K$. That is, F attains its minimum on K.*

Proof: Let $m = \inf\{F(x) : x \in K\}$ and $\{x_n\}$ a sequence in K for which $m = \lim F(x_n)$. If (a) holds, then by weak sequential compactness there is a subsequence of $\{x_n\}$ denoted by $\{x_{n'}\}$ which converges weakly to $x_0 \in V$; Lemma 2.1 shows $x_0 \in K$. The weak lower-semi-continuity of F shows $F(x_0) \leq \liminf F(x_{n'}) = m$, hence, $F(x_0) = m$ and the result follows. For the case of (b), let F have the growth property at $z \in K$ and let $R > 0$ be such that $F(x) > F(z)$ whenever $\|z - x\| \geq R$ and $x \in K$. Then set $B \equiv \{x \in V : \|x - z\| \leq R\}$ and apply (a) to the closed, convex and bounded set $B \cap K$. The result follows from the observation $\inf\{F(x) : x \in K\} = \inf\{F(x) : x \in B \cap K\}$.

We note that if K is bounded then F has the growth property at every point of K; thus the case (b) of Theorem 2.2 includes (a) as a special case. Nevertheless, we prefer to leave Theorem 2.2 in its (possibly) more instructive form as given.

2.2

The condition that a function be weakly lower-semi-continuous is in general difficult to verify. However for those functions which are convex (see below), the lower-semi-continuity is the same for the weak and strong notions; this can be proved directly from Lemma 2.1. We shall consider a class of functions for which convexity and lower semicontinuity are easy to check and, furthermore, this class contains all examples of interest to us here.

Definition. The function $F : K \to \mathbb{R}$ is *convex* if its domain K is convex and for all $x, y \in K$ and $t \in [0, 1]$ we have

$$F(tx + (1-t)y) \leq tF(x) + (1-t)F(y) \ . \tag{2.1}$$

Definition. The function $F : K \to \mathbb{R}$ is *G-differentiable* at $x \in K$ if K is convex and if there is a $F'(x) \in V'$ such that

$$\lim_{t \to 0^+} \frac{1}{t}\Big[F(x + t(y - x)) - F(x)\Big] = F'(x)(y - x)$$

for all $y \in K$. $F'(x)$ is called the *G-differential* of F at x. If F is G-differentiable at every point in K, then $F' : K \to V'$ is the *gradient* of F on K and F is the *potential* of the function F'.

The G-differential $F'(x)$ is precisely the directional derivative of F at the point x in the direction toward y. The following shows how it characterizes convexity of F.

Theorem 2.3 *Let $F : K \to \mathbb{R}$ be G-differentiable on the convex set K. The following are equivalent:* (a) *F is convex,* (b) *For each pair $x, y \in K$ we have*

$$F'(x)(y - x) \leq F(y) - F(x) \ . \tag{2.2}$$

(c) *For each pair $x, y \in K$ we have*

$$(F'(x) - F'(y))(x - y) \geq 0 \ . \tag{2.3}$$

Proof: If F is convex, then $F(x + t(y - x)) \leq F(x) + t(F(y) - F(x))$ for $x, y \in K$ and $t \in [0, 1]$, so (2.2) follows. Thus (a) implies (b). If (b) holds, we obtain $F'(y)(x - y) \leq F(x) - F(y)$ and $F(x) - F(y) \leq F'(x)(x - y)$, so (c) follows.

Finally, we show (c) implies (a). Let $x, y \in K$ and define $\varphi : [0, 1] \to \mathbb{R}$ by

$$\varphi(t) = F(tx + (1 - t)y) = F(y + t(x - y)) \ , \qquad t \in [0, 1] \ .$$

Then $\varphi'(t) = F'(y + t(x - y))(x - y)$ and we have for $0 \leq s < t \leq 1$ the estimate

$$(\varphi'(t) - \varphi'(s))(t - s) = (F'(y + t(x - y)) - F'(y + s(x - y)))((t - s)(x - y)) \geq 0$$

from (c), so φ' is non-decreasing. The Mean-Value Theorem implies that

$$\frac{\varphi(1) - \varphi(t)}{1 - t} \geq \frac{\varphi(t) - \varphi(0)}{t - 0} \ , \qquad 0 < t < 1 \ .$$

Hence, $\varphi(t) \leq t\varphi(1) + (1 - t)\varphi(0)$, and this is just (2.1).

Corollary 2.4 *Let F be G-differentiable and convex. Then F is weakly lower-semi-continuous on K.*

Proof: Let the sequence $\{x_n\} \subset K$ converge weakly to $x \in K$. Since $F'(x) \in V'$, we have $\lim F'(x)(x_n) = F'(x)(x)$, so from (2.2) we obtain

$$\liminf(F(x_n) - F(x)) \geq \liminf F'(x)(x_n - x) = 0 \ .$$

This shows F is weakly lower-semi-continuous at $x \in K$.

Corollary 2.5 *In the situation of Corollary 2.4, for each pair $x, y \in K$ the function*

$$t \longmapsto F'(x + t(y - x))(y - x) \ , \qquad t \in [0, 1]$$

is continuous.

Proof: We need only observe that in the proof of Theorem 2.3 the function φ' is a monotone derivative and thereby must be continuous.

2.3

Our goal is to consider the special case of Theorem 2.2 that results when F is a convex potential function. It will be convenient in the applications to have the hypothesis on F stated in terms of its gradient F'.

Lemma 2.6 *Let F be G-differentiable and convex. Suppose also we have*

$$\lim_{\|x\| \to +\infty} \frac{F'(x)(x)}{\|x\|} = +\infty \ , \qquad x \in K \ .$$

Then $\lim_{\|x\| \to \infty} F(x) = +\infty$, so F has the growth property at every point in K.

Proof: We may assume $\theta \in K$. For each $x \in K$ we obtain from Corollary 2.5

$$\begin{aligned} F(x) - F(\theta) &= \int_0^1 F'(tx)(x)\, dt \\ &= \int_0^1 (F'(tx) - F'(\theta))(x)\, dt + F'(\theta)(x) \ . \end{aligned}$$

With (2.3) this implies

$$F(x) - F(\theta) \geq \int_{1/2}^{1} (F'(tx) - F'(\theta))(x)\,dt + F'(\theta)(x)\ . \tag{2.4}$$

From the Mean-Value Theorem it follows that for some $s = s(x) \in [\frac{1}{2}, 1]$

$$\begin{aligned} F(x) - F(\theta) &\geq \left(\tfrac{1}{2}\right)(F'(sx)(x) + F'(\theta)(x)) \\ &\geq \left(\tfrac{1}{2}\right)\|x\|\left\{\frac{F'(sx)(sx)}{\|sx\|} - \|F'(\theta)\|_{V'}\right\}\ . \end{aligned}$$

Since $\|sx\| \geq (\frac{1}{2})\|x\|$ for all $x \in K$, the result follows.

Definitions. Let D be a non-empty subset of V and $T : D \to V'$ be a function. Then T is *monotone* if

$$(T(x) - T(y))(x - y) \geq 0\ , \qquad x, y \in D\ ,$$

and *strictly monotone* if equality holds only when $x = y$. We call T *coercive* if

$$\lim_{\|x\|\to\infty} \left(\frac{T(x)(x)}{\|x\|}\right) = +\infty\ .$$

After the preceding remarks on potential functions, we have the following fundamental results.

Theorem 2.7 *Let K be a non-empty closed, convex subset of the real separable Hilbert space V, and let the function $F : K \to \mathbb{R}$ be G-differentiable on K. Assume the gradient F' is monotone and either* (a) *K is bounded or* (b) *F' is coercive. Then the set $M \equiv \{x \in K : F(x) \leq F(y)$ for all $y \in K\}$ is non-empty, closed and convex, and $x \in M$ if and only if*

$$x \in K : \quad F'(x)(y - x) \geq 0\ , \qquad y \in K\ . \tag{2.5}$$

Proof: That M is non-empty follows from Theorems 2.2 and 2.3, Corollary 2.4 and Lemma 2.6. Each of the sets $M_y \equiv \{x \in K : F(x) \leq F(y)\}$ is closed and convex so their intersection, M, is closed and convex. If $x \in M$ then (2.5) follows from the definition of $F'(x)$; conversely, (2.2) shows that (2.5) implies $x \in M$.

2.4

We close with a sufficient condition for uniqueness of the minimum point.

Definition. The function $F : K \to \mathbb{R}$ is *strictly convex* if its domain is convex and for $x, y \in K$, $x \neq y$, and $t \in (0,1)$ we have

$$F(tx + (1-t)y) < tF(x) + (1-t)F(y) \ .$$

Theorem 2.8 *A strictly convex function $F : K \to \mathbb{R}$ has at most one point at which the minimum is attained.*

Proof: Suppose $x_1, x_2 \in K$ with $F(x_1) = F(x_2) = \inf\{F(y) : y \in K\}$ and $x_1 \neq x_2$. Since $\frac{1}{2}(x_1 + x_2) \in K$, the strict convexity of F gives

$$F\left((\tfrac{1}{2})(x_1 + x_2)\right) < \left(\tfrac{1}{2}\right)(F(x_1) + F(x_2)) = \inf\{F(y) : y \in K\} \ ,$$

and this is a contradiction.

The third part of the proof of Theorem 2.3 gives the following.

Theorem 2.9 *Let F be G-differentiable on K. If the gradient F' is strictly monotone, then F is strictly convex.*

3 Variational Inequalities

The characterization (2.5) of the minimum point u of F on K is an example of a *variational inequality*. It expresses the fact that from the minimum point the function does not decrease in any direction into the set K. Moreover, if the minimum point is an interior point of K, then we obtain the "variational equality" $F'(u) = 0$, a functional equation for the (gradient) operator F'.

3.1

We shall write out the special form of the preceding results which occur when F is a quadratic function. Thus, V is a real Hilbert space, $f \in V'$, and $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is continuous, bilinear and symmetric. Define $F : V \to \mathbb{R}$ by

$$F(v) = \left(\tfrac{1}{2}\right) a(v,v) - f(v) \ , \qquad v \in V \ . \tag{3.1}$$

From the symmetry of $a(\cdot,\cdot)$ we find the G-differential of F is given by

$$F'(u)(v-u) = a(u,v-u) - f(v-u) \ , \qquad u,v \in V \ .$$

If $\mathcal{A} : V \to V'$ is the operator characterizing the form $a(\cdot,\cdot)$, cf. Section I.5.4, then we obtain

$$F'(u) = \mathcal{A}u - f \ , \qquad u \in V \ . \tag{3.2}$$

To check the convexity of F by the monotonicity of its gradient, we compute

$$(F'u - F'v)(u-v) = a(u-v, u-v) = \mathcal{A}(u-v)(u-v) \ .$$

Thus, F' is monotone (strictly monotone) exactly when $a(\cdot,\cdot)$ is non-negative (respectively, positive), and this is equivalent to $\mathcal{A}$ being monotone (respectively, positive) (cf. Section V.1). The growth of F is implied by the statement

$$\lim_{\|v\|\to\infty} \left(\frac{a(v,v)}{\|v\|} \right) = +\infty \ . \tag{3.3}$$

Since $F(v) \geq (\frac{1}{2})a(v,v) - \|f\| \cdot \|v\|$, from the identity (3.2) we find that (3.3) is equivalent to F' being coercive.

The preceding remarks show that Theorems 2.7 and 2.8 give the following.

Theorem 3.1 *Let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ be continuous, bilinear, symmetric and non-negative. Suppose $f \in V'$ and K is a closed convex subset of V. Assume either* (a) *K is bounded or* (b) *$a(\cdot,\cdot)$ is V-coercive. Then there exists a solution of*

$$u \in K : \quad a(u, v-u) \geq f(v-u) \ , \qquad v \in K \ . \tag{3.4}$$

There is exactly one such u in the case of (b)*; there is exactly one in case* (a) *if we further assume $a(\cdot,\cdot)$ is positive.*

Finally we note that when K is the whole space V, then (3.4) is equivalent to

$$u \in V : \quad a(u,v) = f(v) \ , \qquad v \in V \ , \tag{3.5}$$

the generalized boundary value problem studied in Chapter III. For this reason, when (3.5) is equivalent to a boundary value problem, (3.5) is called the *variational form* of that problem and such problems are called *variational boundary value problems*.

We shall illustrate some very simple variational inequalities by examples in which we characterize the solution by other means.

3.2 Projection

Given the Hilbert space V, the closed convex subset K, and the point $u_0 \in V$, we define

$$a(u,v) = (u,v)_V \ , \quad f(v) = (u_0,v)_V \ , \qquad u,v \in V \ .$$

Then (3.1) gives the function

$$F(v) = \left(\tfrac{1}{2}\right)\left\{\|u_0 - v\|^2 - \|u_0\|^2\right\} \ , \qquad v \in V \ ,$$

so $u \in K$ is the minimum of F on K if and only if

$$\|u_0 - u\| \leq \|u_0 - v\| \ , \qquad v \in K \ .$$

That is, u is that (unique) point of K which is closest to u_0. The existence and uniqueness follows from Theorem 3.1; in this case we have the equivalent of Theorem I.4.3. The computation

$$F'(u)(v-u) = (u - v_0, v - u)_V$$

shows that u is characterized by the variational inequality

$$u \in K : \quad (u - u_0, v - u)_V \geq 0 \ , \qquad v \in K \ ,$$

and the geometric meaning of this inequality is that the angle between $u - u_0$ and $v - u$ is between $-\pi/2$ and $\pi/2$ for each $v \in K$. If K is a subspace of V, this is equivalent to (3.5) which says $u - u_0$ is orthogonal to K. That is, u is the projection of u_0 on the space K, cf. Section I.4.3.

3.3 Dirichlet's Principle

Let G be a bounded open set in $\mathbb{R}^n$ and $V = H_0^1(G)$. Let $F \in L^2(\Omega)$ and define

$$a(u,v) = \int_G \nabla u \cdot \nabla v \, dx \ , \quad f(v) = \int_G F(x)v(x)\,dx \ , \qquad u,v \in V \ .$$

Thus, the function to be minimized is

$$E(v) = \left(\tfrac{1}{2}\right)\int_G \sum_{j=1}^n |\partial_j v|^2 \, dx - \int_G Fv \, dx \ , \qquad v \in V \ .$$

In the applications this is a measure of the "energy" in the system. Take K to be the whole space: $K = V$. The point u at which E attains its minimum is characterized by (3.5). Thus, the solution is characterized by the Dirichlet problem (1.1), cf. Chapter III.

3.4 Free Boundary Problem

We take the space V, the form $a(\cdot,\cdot)$ and functional f as above. Let $g \in H_0^1(\Omega)$ and define

$$K = \{v \in H_0^1(G) : v(x) \geq g(x) \quad \text{a.e.} \quad x \in G\} \ .$$

Since $a(\cdot,\cdot)$ is V-coercive, there exists a unique solution u of (3.4). This solution is characterized by the following:

$$\left.\begin{aligned} & u \geq g \text{ in } G \ , \quad u = 0 \text{ on } \partial G \ , \\ & -\Delta_n u - F \geq 0 \text{ in } G \ , \quad \text{and} \\ & (u-g)(-\Delta_n u - F) = 0 \text{ in } G \ . \end{aligned}\right\} \tag{3.6}$$

The first follows from $u \in K$ and the second is obtained from (3.4) by setting $v = u + \varphi$ for any $\varphi \in C_0^\infty(G)$ with $\varphi \geq 0$. Given the first two lines of (3.6), the third line follows by setting $v = g$ in (3.4). One can show, conversely, that any $u \in H^1(G)$ satisfying (3.6) is the solution of (3.4). Note that the region G is partitioned into two parts

$$G_0 = \{x : u(x) = g(x)\} \quad , \quad G_+ = \{x : u(x) > g(x)\}$$

and $-\Delta_n u = F$ in G_+. That is, in G_0 (G_+) the first (respectively, second) inequality in (3.6) is replaced by the corresponding equation. There is a *free boundary* at the interface between G_0 and G_+; locating this free boundary is equivalent to reducing (3.6) to a Dirichlet problem.

3.5 Unilateral Boundary Condition

Choose $V = H^1(G)$ and $K = \{v \in V : v \geq g_1 \text{ on } \partial G\}$, where $g_1 \in H^1(G)$ is given. Let $F(\cdot) \in L^2(G)$, $g_2 \in L^2(\partial G)$ and define $f \in V'$ by

$$f(v) = \int_G Fv\,dx + \int_{\partial G} g_2 v\,ds \ , \qquad v \in V$$

where we suppress the trace operator in the above and hereafter. Set $a(u,v) = (u,v)_{H^1(G)}$. Theorem 3.1 shows there exists a unique solution

$u \in K$ of (3.4). This solution is characterized by the following:

$$\left.\begin{aligned} &-\Delta_n u + u = F \text{ in } G \ , \\ &u \geq g_1 \text{ on } \partial G \ , \\ &\frac{\partial u}{\partial \nu} \geq g_2 \text{ on } \partial G \ , \quad \text{and} \\ &\left(\frac{\partial u}{\partial \nu} - g_2\right)(u - g_1) = 0 \text{ on } \partial G \ . \end{aligned}\right\} \tag{3.7}$$

We shall show that the solution of (3.4) satisfies (3.7); the converse is left to an exercise. The first inequality in (3.7) follows from $u \in K$. If $\varphi \in C_0^\infty(G)$, then setting $v = u + \varphi$, then $v = u - \varphi$ in (3.4) we obtain the partial differential equation in (3.7). Inserting this equation in (3.4) and using the abstract Green's formula (Theorem III.2.3), we obtain

$$\int_{\partial G} \frac{\partial u}{\partial \nu}(v - u)\, ds \geq \int_{\partial G} g_2(v - u) \ , \qquad v \in K \ . \tag{3.8}$$

If $w \in H^1(G)$ satisfies $w \geq 0$ on ∂G, we may set $v = u + w$ in (3.8); this gives the second inequality in (3.7). Setting $v = g_1$ in (3.8) yields the last equation in (3.7). Note that there is a region Γ_0 in ∂G on which $u = g_1$, and $\partial u / \partial \nu = g_2$ on $\partial G \sim \Gamma_0$. Thus, finding u is equivalent to finding Γ_0, so we may think of (3.7) as another free boundary problem.

4 Optimal Control of Boundary Value Problems

4.1

Various optimal control problems are naturally formulated as minimization problems like those of Section 2. We illustrate the situation with a model problem which we discuss in this section.

Example. Let G be a bounded open set in $\mathbb{R}^n$ whose boundary ∂G is a C^1-manifold with G on one side. Let $F \in L^2(G)$ and $g \in L^2(\partial G)$ be given. Then for each *control* $v \in L^2(\partial G)$ there is a corresponding *state* $y \in H^1(G)$ obtained as the unique solution of the *system*

$$\left.\begin{aligned} &-\Delta_n y + y = F && \text{in } G \\ &\frac{\partial y}{\partial \nu} = g + v && \text{on } \partial G \end{aligned}\right\} \tag{4.1}$$

and we denote the dependence of y on v by $y = y(v)$. Assume that we may observe the state y only on ∂G and that our objective is to choose v so as to place the *observation* $y(v)|_{\partial G}$ closest to a given desired observation $w \in L^2(\partial G)$. Each control v is exerted at some *cost*, so the optimal control problem is to minimize the "error plus cost"

$$J(v) = \int_{\partial G} |y(v) - w|^2 \, dx + c \int_{\partial G} |v|^2 \, dx \tag{4.2}$$

over some given set of *admissible controls* in $L^2(\partial G)$. An admissible control u at which J attains its minimum is called an *optimal control*. We shall briefly consider problems of existence or uniqueness of optimal controls and alternate characterizations of them, and then apply these general results to our model problem.

We shall formulate the model problem (4.1), (4.2) in an abstract setting suggested by Chapter III. Thus, let V and H be real Hilbert spaces with V dense and continuously imbedded in H; identify the pivot space H with its dual and thereby obtain the inclusions $V \hookrightarrow H \hookrightarrow V'$. Let $a(\cdot,\cdot)$ be a continuous, bilinear and coercive form on V for which the corresponding operator $\mathcal{A} : V \to V'$ given by

$$a(u,v) = \mathcal{A}u(v) \ , \qquad u, v \in V$$

is necessarily a continuous bijection with continuous inverse. Finally, let $f \in V'$ be given. (The system (4.1) with $v \equiv 0$ can be obtained as the operator equation $\mathcal{A}y = f$ for appropriate choices of the preceding data; cf. Section III.4.2 and below.)

To obtain a control problem we specify in addition to the state space V and data space V' a Hilbert space U of controls and an operator $\mathcal{B} \in \mathcal{L}(U, V')$. Then for each control $v \in U$, the corresponding state $y = y(v)$ is the solution of the system (cf. (4.1))

$$\mathcal{A}y = f + \mathcal{B}v \ , \qquad y = y(v) \ . \tag{4.3}$$

We are given a Hilbert space W of observations and an operator $\mathcal{C} \in \mathcal{L}(V, W)$. For each state $y \in V$ there is a corresponding observation $\mathcal{C}y \in W$ which we want to force close to a given desired observation $w \in W$. The cost of applying the control $v \in U$ is given by $Nv(v)$ where $N \in \mathcal{L}(U, U')$ is symmetric and monotone. Thus, to each control $v \in U$ there is the "error plus cost" given by

$$J(v) \equiv \|\mathcal{C}y(v) - w\|_W^2 + Nv(v) \ . \tag{4.4}$$

The *optimal control problem* is to minimize (4.4) over a given non-empty closed convex subset U_{ad} of *admissible controls* in U. An *optimal control* is a solution of

$$u \in U_{\mathrm{ad}} : J(u) \leq J(v) \quad \text{for all} \quad v \in U_{\mathrm{ad}} \ . \tag{4.5}$$

4.2

Our objectives are to give sufficient conditions for the existence (and possible uniqueness) of optimal controls and to characterize them in a form which gives more information.

We shall use Theorem 2.7 to attain these objectives. In order to compute the G-differential of J we first obtain from (4.3) the identity

$$\mathcal{C}y(v) - w = \mathcal{C}\mathcal{A}^{-1}\mathcal{B}v + \mathcal{C}\mathcal{A}^{-1}f - w$$

which we use to write (4.4) in the form

$$J(v) = \|\mathcal{C}\mathcal{A}^{-1}\mathcal{B}v\|_W^2 + Nv(v) + 2(\mathcal{C}\mathcal{A}^{-1}\mathcal{B}v, \mathcal{C}\mathcal{A}^{-1}f - w)_W + \|\mathcal{C}\mathcal{A}^{-1}f - w\|_W^2 \ .$$

Having expressed J as the sum of quadratic, linear and constant terms, we easily obtain the G-differential

$$\begin{aligned} J'(v)(\varphi) &= 2\Big\{(\mathcal{C}\mathcal{A}^{-1}\mathcal{B}v, \mathcal{C}\mathcal{A}^{-1}\mathcal{B}\varphi)_W \\ &\qquad + Nv(\varphi) + (\mathcal{C}\mathcal{A}^{-1}\mathcal{B}\varphi, \mathcal{C}\mathcal{A}^{-1}f - w)_W\Big\} \\ &= 2\Big\{(\mathcal{C}y(v) - w, \mathcal{C}\mathcal{A}^{-1}\mathcal{B}\varphi)_W + Nv(\varphi)\Big\} \ . \end{aligned} \tag{4.6}$$

Thus, we find that the gradient J' is monotone and

$$\left(\tfrac{1}{2}\right) J'(v)(v) \geq Nv(v) - (\text{const.})\|v\|_U \ ,$$

so J' is coercive if N is coercive, i.e., if

$$Nv(v) \geq c\|v\|_U^2 \ , \qquad v \in U_{\mathrm{ad}} \ , \tag{4.7}$$

for some $c > 0$. Thus, we obtain from Theorem 2.7 the following.

Theorem 4.1 *Let the optimal control problem be given as in Section 4.1. That is, we are to minimize* (4.4) *subject to* (4.3) *over the non-empty closed convex set* U_{ad}. *Then if either* (a) U_{ad} *is bounded or* (b) N *is coercive over* U_{ad}, *then the set of optimal controls is non-empty, closed and convex.*

Corollary 4.2 *In case* (b) *there is a unique optimal control.*

Proof: This follows from Theorem 2.9 since (4.7) implies J' is strictly monotone.

4.3

We shall characterize the optimal controls by variational inequalities. Thus, u is an optimal control if and only if

$$u \in U_{\mathrm{ad}} : J'(u)(v-u) \geq 0 \ , \qquad v \in U_{\mathrm{ad}} \ ; \tag{4.8}$$

this is just (2.5). This variational inequality is given by (4.6), of course, but the resulting form is difficult to interpret. The difficulty is that it compares elements of the observation space W with those of the control space U; we shall obtain an equivalent characterization which contains a variational inequality only in the control space U. In order to convert the first term on the right side of (4.6) into a more convenient form, we shall use the Riesz map R_W of W onto W' given by (cf. Section I.4.3)

$$R_W(x)(y) = (x,y)_W \ , \qquad x, y \in W$$

and the dual $\mathcal{C}' \in \mathcal{L}(W', V')$ of $\mathcal{C}$ given by (cf. Section I.5.1)

$$\mathcal{C}'(f)(x) = f(\mathcal{C}(x)) \ , \qquad f \in W' \ , \ x \in V \ .$$

Then from (4.6) we obtain

$$\begin{aligned} \left(\tfrac{1}{2}\right) J'(u)(v) &= (\mathcal{C}y(u) - w, \mathcal{C}\mathcal{A}^{-1}\mathcal{B}v)_W + Nu(v) \\ &= R_W(\mathcal{C}y(u) - w)(\mathcal{C}\mathcal{A}^{-1}\mathcal{B}v) + Nu(v) \\ &= \mathcal{C}' R_W(\mathcal{C}y(u) - w)(\mathcal{A}^{-1}\mathcal{B}v) + Nu(v) \ , \qquad u, v \in U \ . \end{aligned}$$

To continue we shall need the dual operator $\mathcal{A}' \in \mathcal{L}(V, V')$ given by

$$\mathcal{A}'x(y) = \mathcal{A}y(x) \ , \qquad x, y \in V \ ,$$

where V'' is naturally identified with V. Since $\mathcal{A}'$ arises from the bilinear form adjoint to $a(\cdot,\cdot)$, $\mathcal{A}'$ is an isomorphism. Thus, for each control $v \in U$ we can define the corresponding *adjoint state* $p = p(v)$ as the unique solution of the system

$$\mathcal{A}'p = \mathcal{C}'R_W(\mathcal{C}y(v) - w) \ , \qquad p = p(v) \ . \tag{4.9}$$

From above we then have

$$\begin{aligned} \left(\tfrac{1}{2}\right) J'(u)(v) &= \mathcal{A}'p(u)(\mathcal{A}^{-1}\mathcal{B}v) + Nu(v) \\ &= \mathcal{B}v(p(u)) + Nu(v) \\ &= \mathcal{B}'p(u)(v) + Nu(v) \end{aligned}$$

where $\mathcal{B}' \in \mathcal{L}(V, U')$ is the indicated dual operator. These computations lead to a formulation of (4.8) which we summarize as follows.

Theorem 4.3 *Let the optimal control problem be given as in* (4.1). *Then a necessary and sufficient condition for u to be an optimal control is that it satisfy the following system:*

$$\left.\begin{aligned} & u \in U_{\mathrm{ad}} \ , \quad \mathcal{A}y(u) = f + \mathcal{B}u \ , \\ & \mathcal{A}'p(u) = \mathcal{C}'R_W(\mathcal{C}y(u) - w) \ , \\ & (\mathcal{B}'p(u) + Nu)(v - u) \geq 0 \ , \quad \textit{all} \ \ v \in U_{\mathrm{ad}} \ . \end{aligned}\right\} \tag{4.10}$$

The system (4.10) is called the *optimality system* for the control problem. We leave it as an exercise to show that a solution of the optimality system satisfies (4.8).

4.4

We shall recover the Example of Section 4.1 from the abstract situation above. Thus, we choose $V = H^1(G)$, $a(u, v) = (u, v)_{H^1(G)}$, $U = L^2(\partial G)$ and define

$$\begin{aligned} f(v) &= \int_G F(x)v(x)\,dx + \int_{\partial G} g(s)v(s)\,ds \ , & v \in V \ , \\ \mathcal{B}u(v) &= \int_{\partial G} u(s)v(s)\,ds \ , & u \in U \ , \ v \in V \ . \end{aligned}$$

The state $y(u)$ of the system determined by the control u is given by (4.3), i.e.,

$$\begin{aligned} -\Delta_n y + y = F \quad \text{in} \quad G \ , \\ \frac{\partial y}{\partial \nu} = g + u \quad \text{on} \quad \partial G \ . \end{aligned} \tag{4.11}$$

Choose $W = L^2(\partial G)$, $w \in W$, and define

$$
\begin{aligned}
Nu(v) &= c \int_{\partial G} u(s)v(s)\,ds\ , && u, v \in W\ ,\ (c \geq 0) \\
\mathcal{C}u(v) &\equiv \int_{\partial G} u(s)v(s)\,ds\ , && u \in V\ ,\ v \in W\ .
\end{aligned}
$$

The adjoint state equation (4.9) becomes

$$
\begin{aligned}
-\Delta_n p + p = 0 \quad &\text{in} \quad G \\
\frac{\partial p}{\partial \nu} = y - w \quad &\text{on} \quad \partial G
\end{aligned}
\tag{4.12}
$$

and the variational inequality is given by

$$
u \in U_{\text{ad}} : \int_{\partial G} (p + cu)(v - u)\,ds \geq 0\ , \qquad v \in U_{\text{ad}}\ . \tag{4.13}
$$

From Theorem 4.1 we obtain the existence of an optimal control if U_{ad} is bounded or if $c > 0$. Note that

$$
J(v) = \int_{\partial G} |y(v) - w|^2\,ds + c \int_{\partial G} |v|^2\,ds \tag{4.14}
$$

is strictly convex in either case, so uniqueness follows in both situations. Theorem 4.3 shows the unique optimal control u is characterized by the optimality system (4.11), (4.12), (4.13). We illustrate the use of this system in two cases.

4.5 $U_{\text{ad}} = L^2(\partial G)$

This is the case of *no constraints* on the control. Existence of an optimal control follows if $c > 0$. Then (4.13) is equivalent to $p + cu = 0$. The optimality system is equivalent to

$$
\begin{aligned}
-\Delta_n y + y &= F\ , & -\Delta_n p + p &= 0 \quad \text{in } G \\
\frac{\partial y}{\partial \nu} &= g - \left(\frac{1}{c}\right) p\ , & \frac{\partial p}{\partial \nu} &= y - w \quad \text{on} \quad \partial G
\end{aligned}
$$

and the optimal control is given by $u = -(1/c)p$.

Consider the preceding case with $c = 0$. We show that an optimal control might not exist. First show $\inf\{J(v) : v \in U\} = 0$. Pick a sequence $\{w_m\}$ of very smooth functions on ∂G such that $w_m \to w$ in $L^2(\partial G)$. Define y_m by

$$
\begin{aligned}
-\Delta_n y_m + y_m &= F \quad \text{in} \quad G \\
y_m &= w_m \quad \text{on} \quad \partial G
\end{aligned}
$$

and set $v_m = (\partial y_m / \partial \nu) - g$, $m \geq 1$. Then $v_m \in L^2(\partial G)$ and $J(v_m) = \|w_m - w\|^2_{L^2(\partial G)} \to 0$. Second, note that if u is an optimal control, then $J(u) = 0$ and the corresponding state y satisfies

$$
\begin{aligned}
-\Delta_n y + y &= F \quad \text{in} \quad G \\
y &= w \quad \text{on} \quad \partial G \ .
\end{aligned}
$$

Then we have (formally) $u = (\partial y / \partial \nu) - g$. However, if $w \in L^2(\partial G)$ one does not in general have $(\partial y / \partial \nu) \in L^2(\partial G)$. Thus u might not be in $L^2(\partial G)$ in which case there is no optimal control (in $L^2(\partial G)$).

4.6

$U_{\text{ad}} = \{v \in L^2(\partial G) : 0 \leq v(s) \leq M \text{ a.e.}\}$. Since the set of admissible controls is bounded, there exists a unique optimal control u characterized by the optimality system (4.10). Thus, u is characterized by (4.11), (4.12) and

$$
\begin{aligned}
&\text{if} \quad 0 < u < M \ , \quad \text{then} \quad p + cu = 0 \\
&\text{if} \quad u = 0 \ , \quad \text{then} \quad p \geq 0 \ , \ \text{and} \\
&\text{if} \quad u = M \ , \quad \text{then} \quad p + cu \leq 0 \ .
\end{aligned}
\tag{4.15}
$$

We need only to check that (4.13) and (4.15) are equivalent. The boundary is partitioned into the three regions determined by the three respective cases in (4.15). This is analogous to the free boundary problems encountered in Sections 3.3 and 3.4.

We specialize the above to the case of "free control," i.e., $c = 0$. One may search for an optimal control in the following manner. Motivated by (4.11) and (4.14), we consider the solution Y of the Dirichlet problem

$$
\begin{aligned}
-\Delta_n Y + Y &= F \quad \text{in} \quad G \ , \\
Y &= w \quad \text{on} \quad \partial G \ .
\end{aligned}
$$

If it happens that

$$0 \leq \frac{\partial Y}{\partial \nu} - g \leq M \quad \text{on} \quad \partial G \ , \tag{4.16}$$

then the optimal control is given by (4.11) as

$$u = \frac{\partial Y}{\partial \nu} - g \ .$$

Note that $u \in U_{\text{ad}}$ and $J(u) = 0$.

We consider the contrary situation in which (4.16) does not hold. Specifically we shall show that (when all aspects of the problem are regular) the set $\Gamma \equiv \{s \in \partial G : 0 < u(s) < M,\, p(s) = 0\}$ is empty. This implies that the control takes on only its extreme values $0, M$; this is a result of "bang-bang" type.

Partition Γ into the three parts $\Gamma_0 = \{s \in \Gamma : y(s) = w(s)\}$, $\Gamma_+ = \{s \in \Gamma : y(s) > w(s)\}$ and $\Gamma_- = \{s \in \Gamma : y(s) < w(s)\}$. On any interval in Γ_0 we have $p = 0$ (by definition of Γ) and $\frac{\partial p}{\partial \nu} = 0$ from (4.12). From the uniqueness of the Cauchy problem for the elliptic equation in (4.12), we obtain $p = 0$ in G, hence, $y = w$ on ∂G. But this implies $y = Y$, hence (4.16) holds. This contradiction shows Γ_0 is empty. On any interval in Γ_+ we have $p = 0$ and $\frac{\partial p}{\partial \nu} > 0$. Thus, $p < 0$ in some neighborhood (in $\bar{G}$) of that interval. But $\Delta p < 0$ in the neighborhood follows from (4.12), so a maximum principle implies $\frac{\partial p}{\partial \nu} \leq 0$ on that interval. This contradiction shows Γ_+ is empty. A similar argument holds for Γ_- and the desired result follows.

5 Approximation of Elliptic Problems

We shall discuss the *Rayleigh-Ritz-Galerkin* procedure for approximating the solution of an elliptic boundary value problem. This procedure can be motivated by the situation of Section 3.1 where the abstract boundary value problem (3.5) is known to be equivalent to minimizing a quadratic function (3.1) over the Hilbert space V. The procedure is to pick a closed subspace S of V and minimize the quadratic function over S. This is the Rayleigh-Ritz method. The resulting solution is close to the original solution if S closely approximates V. The approximate solution is characterized by the abstract boundary vlaue problem obtained by replacing V with S; this gives the (equivalent) Galerkin method of obtaining an approximate solution. The very important *finite-element method* consists of the above procedure applied with a space S of piecewise polynomial functions which approximates the whole space V. The resulting finite-dimensional problem can be solved efficiently by computers. Our objectives are to describe the Rayleigh-Ritz-Galerkin procedure, obtain estimates on the error that results from the approximation, and then to give some typical convergence rates that result from standard finite-element or *spline* approximations of the space. We shall also construct some of these approximating subspaces and prove the error estimates as an application of the minimization theory of Section 2.

5.1

Suppose we are given an abstract boundary value problem: V is a Hilbert space, $a(\cdot,\cdot) : V \times V \to \mathbb{K}$ is continuous and sesquilinear, and $f \in V'$. The problem is to find u satisfying

$$u \in V : a(u,v) = f(v) \ , \qquad v \in V \ . \tag{5.1}$$

Let S be a subspace of V. Then we may consider the related problem of determining u_s satisfying

$$u_s \in S : a(u_s,v) = f(v) \ , \qquad v \in S \ . \tag{5.2}$$

We shall show that the error $u_s - u$ is small if S approximates V sufficiently well.

Theorem 5.1 *Let $a(\cdot,\cdot)$ be a V-coercive continuous sesquilinear form and $f \in V'$. Let S be a closed subspace of V. Then* (5.1) *has a unique solution u and* (5.2) *has a unique solution u_s. Furthermore we have the estimate*

$$\|u_s - u\| \le (K/c) \inf\{\|u - v\| : v \in S\} \ , \tag{5.3}$$

where K is the bound on $a(\cdot,\cdot)$ (cf. the inequality I.(5.2)*) and c is the coercivity constant (cf. the inequality* III.(2.3)*).*

Proof: The existence and uniqueness of the solutions u and u_s of (5.1) and (5.2) follow immediately from Theorem III.2.1 or Theorem 3.1, so we need only to verify the estimate (5.3). By subtracting (5.1) from (5.2) we obtain

$$a(u_s - u, v) = 0 \ , \qquad v \in S \ . \tag{5.4}$$

Thus for any $w \in S$ we have

$$a(u_s - u, u_s - u) = a(u_s - u, w - u) + a(u_s - u, u_s - w) .$$

Since $u_s - w \equiv v \in S$ it follows that the last term is zero, so we obtain

$$c\|u_s - u\|^2 \le K\|u_s - u\| \, \|w - u\| , \qquad w \in S .$$

This gives the desired result.

Consider for the moment the case of V being separable. Thus, there is a sequence $\{v_1, v_2, v_3, \ldots\}$ in V which is a basis for V. For each integer $m \ge 1$, the set $\{v_1, v_2, \ldots, v_m\}$ is linearly independent and its linear span will be denoted by V_m. If P_m is the projection of V into V_m, then $\lim_{m\to\infty} P_m v = v$ for all $v \in V$. The problem (5.2) with $S = V_m$ is equivalent to

$$u_m \in V_m : a(u_m, v_j) = f(v_j) , \qquad 1 \le j \le m .$$

There is exactly one such u_m for each integer $m \ge 1$ and we have the estimates $\|u_m - u\| \le (K/c)\|u - P_m u\|$. Hence, $\lim_{m\to\infty} u_m = u$ in V and the rate of convergence is determined by that of $\{P_m u\}$ to the solution u of (5.1). Thus we are led to consider an approximating finite-dimensional problem. Specifically u_m is determined by the point $x = (x_1, x_2, \ldots, x_m) \in \mathbb{K}^m$ through the identity $u_m = \sum_{i=1}^m x_i v_i$, and (5.2) is equivalent to the $m \times m$ system of linear equations

$$\sum_{i=1}^{m} a(v_i, v_j) x_i = f(v_j) , \qquad 1 \le j \le m . \tag{5.5}$$

Since $a(\cdot,\cdot)$ is V-coercive, the $m \times m$ coefficient matrix $(a(v_i, v_j))$ is invertible and the linear system (5.5) can be solved for x. The dimension of the system is frequently of the order $m = 10^2$ or 10^3, so the actual computation of the solution may be a non-trivial consideration. It is helpful to choose the basis functions so that most of the coefficients are zero. Thus, the matrix is *sparse* and various special techniques are available for efficiently solving the large linear system. This sparseness of the coefficient matrix is one of the computational advantages of using finite-element spaces. A very special example will be given in Section 5.4 below.

5.2

The fundamental estimate (5.3) is a bound on the error in the norm of the Hilbert space V. In applications to elliptic boundary value problems this corresponds to an *energy estimate*. We shall estimate the error in the norm of a pivot space H. Since this norm is weaker we expect an improvement on the rate of convergence with respect to the approximation of V by S.

Theorem 5.2 *Let $a(\cdot,\cdot)$ be a continuous, sesquilinear and coercive form on the Hilbert space V, and let H be a Hilbert space identified with its dual and in which V is dense, and continuously imbedded. Thus, $V \hookrightarrow H \hookrightarrow V'$. Let $A^* : D^* \to H$ be the operator on H which is determined by the adjoint sesquilinear form, i.e.,*

$$\overline{a(v,w)} = (A^*w, v)_H \ , \qquad w \in D^* \ , \ v \in V$$

(cf. Section III.7.5). Let S be a closed subspace of V and $e^*(S)$ a corresponding constant for which we have

$$\inf\{\|w - v\| : v \in S\} \le e^*(S)|A^*w|_H \ , \qquad w \in D^* \ . \tag{5.6}$$

Then the solutions u of (5.1) and u_s of (5.2) satisfy the estimate

$$|u - u_s|_H \le (K^2/c) \inf\{\|u - v\| : v \in S\} e^*(S) \ . \tag{5.7}$$

Proof: We may assume $u \ne u_s$; define $g = (u - u_s)/|u - u_s|_H$ and choose $w \in D^*$ so that $A^*w = g$. That is,

$$a(v,w) = (v,g)_H \ , \qquad v \in V \ ,$$

and this implies that

$$a(u - u_s, w) = (u - u_s, g)_H = |u - u_s|_H \ .$$

From this identity and (5.4) we obtain for any $v \in S$

$$|u-u_s|_H = a(u-u_s, w-v) \le K\|u-u_s\| \, \|w-v\| \le K\|u-u_s\| e^*(S)|A^*w|_H \ .$$

Since $|A^*w|_H = |g|_H = 1$, the estimate (5.7) follows from (5.3).

Corollary 5.3 *Let $A : D \to H$ be the operator on H determined by $a(\cdot,\cdot)$, V, H, i.e.,*

$$a(w,v) = (Aw,v)_H \ , \qquad w \in D \ , \ v \in V \ .$$

Let $e(S)$ be a constant for which

$$\inf\{\|w-v\| : v \in S\} \leq e(S)|Aw|_H \ , \qquad w \in D \ .$$

Then the solutions of (5.1) *and* (5.2) *satisfy the estimate*

$$|u-u_s|_H \leq (K^2/c)e(S)e^*(S)|Au|_H \ . \tag{5.8}$$

The estimate (5.7) will provide the rate of convergence of the error that is faster than that of (5.3). The added factor $e^*(S)$ arising in (5.6) will depend on how well S approximates the subspace D^* of "smoother" or "more regular" elements of V.

5.3

We shall combine the estimates (5.3) and (5.7) with approximation results that are typical of finite-element or spline function subspaces of $H^1(G)$. This will result in rate of convergence estimates in terms of a parameter $h > 0$ related to mesh size in the approximation scheme. The *approximation assumption* that we make is as follows: Suppose $\mathcal{H}$ is a set of positive numbers, M and $k \geq 0$ are integers, and $\mathcal{S} \equiv \{S_h : h \in \mathcal{H}\}$ is a collection of closed subspaces of $V \subset H^1(G)$ such that

$$\inf\{\|w-v\|_{H^1(G)} : v \in S_h\} \leq Mh^{j-1}\|w\|_{H^j(G)} \tag{5.9}$$

for all $h \in \mathcal{H}$, $1 \leq j \leq k+2$, and $w \in H^j(G) \cap V$. The integer $k+1$ is called the *degree* of $\mathcal{S}$.

Theorem 5.4 *Let V be a closed subspace of $H^1(G)$ with $H_0^1(G) \subset V$ and let $a(\cdot,\cdot) : V \times V \to \mathbb{K}$ be continuous, sesquilinear and V-coercive. Let $\mathcal{S}$ be a collection of closed subspaces of V satisfying* (5.9) *for some $k \geq 0$, and assume $a(\cdot,\cdot)$ is k-regular on V. Let $F \in H^k(G)$ and define $f \in V'$ by $f(v) = (F,v)_H$, $v \in V$, where $H \equiv L^2(G)$. Let u be the solution of* (5.1) *and, for each $h \in \mathcal{H}$, u_h be the solution of* (5.2) *with $S = S_h$. Then for some constant c_1 we have*

$$\|u-u_h\|_{H^1(G)} \leq c_1 h^{k+1} \ , \qquad h \in \mathcal{H} \ . \tag{5.10}$$

If in addition the sesquilinear form adjoint to $a(\cdot,\cdot)$ *is* 0*-regular, then for some constant* c_2 *we have*

$$\|u - u_h\|_{L^2(G)} \le c_2 h^{k+2}\ , \qquad h \in \mathcal{H}\ . \tag{5.11}$$

Proof: Since $F \in H^k(G)$ and $a(\cdot,\cdot)$ is k-regular it follows that $u \in H^{k+2}(G)$. Hence we combine (5.3) with (5.9) to obtain (5.10). If the adjoint form is 0-regular, then in Theorem 5.2 we have $D^* \subset H^2(G)$ and $\|w\|_{H^2(G)} \le$ (const.)$\|A^*w\|_{L^2(G)}$. Hence (5.9) with $j = 2$ gives (5.6) with $e^*(S_h) =$ (const.)h. Thus (5.11) follows from (5.7).

Sufficient conditions for $a(\cdot,\cdot)$ to be k-regular were given in Section III.6. Note that this permits all the hypotheses in Theorem 5.4 to be placed on the *data* in the problem (5.1) being solved. For problems for which appropriate regularity results are not available, one may of course assume the appropriate smoothness of the solution.

5.4

Let G be the interval $(0,1)$ and V a closed subspace of $H^1(G)$. Any function $f \in V$ can be approximated by a piecewise-linear $f_0 \in V$; we need only to choose f_0 so that it agrees with f at the endpoints of the intervals on which f_0 is affine. This is a simple *Lagrange interpolation* of f by the linear finite-element function f_0, and it leads to a family of approximating subspaces of degree 1. We shall describe the spaces and prove the estimates (5.9) for this example.

Let $P = \{0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1\}$ be a partition of G and denote by $\mu(P)$ the mesh of $P : \mu(P) = \max\{x_{j+1} - x_j : 0 \le j \le N\}$. The closed convex set $K = \{v \in V : v(x_j) = 0,\ 0 \le j \le N+1\}$ is basic to our construction. Let $f \in V$ be given and consider the function $F(v) = (\frac{1}{2})|\partial(v - f)|_H^2$ on V, where $H = L^2(G)$. The G-differential is given by

$$F'(u)(v) = (\partial(u - f), \partial v)_H\ , \qquad u, v \in V\ .$$

We easily check that F' is strictly monotone on K; this follows from Theorem II.2.4. Similarly the estimate

$$F'(v)(v) = |\partial v|_H^2 - (\partial f, \partial v)_H \ge |\partial v|_H^2 - |\partial f|_H |\partial v|_H\ , \qquad v \in V\ ,$$

shows F' is coercive on K. It follows from Theorems 2.7 and 2.9 that there is a unique $u_f \in K$ at which F takes its minimal value on K, and it is characterized in (2.5) by

$$u_f \in K: \quad (\partial(u_f - f), \partial v)_H = 0 \ , \qquad v \in K \ .$$

This shows that for each $f \in V$, there exists exactly one $f_0 \in V$ which satisfies

$$f_0 - f \in K \ , \quad (\partial f_0, \partial v)_H = 0 \ , \qquad v \in K \ . \tag{5.12}$$

(They are clearly related by $f_0 = f - u_f$.) The second part of (5.12) states that $-\partial^2 f_0 = 0$ in each subinterval of the partition so f_0 is affine on each subinterval. The first part of (5.12) determines the value of f_0 at each of the points of the partition, so it follows that f_0 is that function in V which is affine in the intervals of P and equals f at the points of P. This f_0 is the linear finite-element interpolant of f.

To compute the error in this interpolation procedure, we first note that

$$|\partial f_0|_H^2 + |\partial(f_0 - f)|_H^2 = |\partial f|_H^2$$

follows from setting $v = f_0 - f$ in (5.12). Thus we obtain the estimate

$$|\partial(f_0 - f)|_H \le |\partial f|_H \ .$$

If $g = f_0 - f$, then from Theorem II.2.4 we have

$$\int_{x_j}^{x_{j+1}} |g|^2 \, dx \le 4\mu(P)^2 \int_{x_j}^{x_{j+1}} |\partial g|^2 \, dx \ , \qquad 0 \le j \le N \ ,$$

and summing these up gives

$$|f - f_0|_H \le 2\mu(P) |\partial(f_0 - f)|_H \ . \tag{5.13}$$

This proves the first two estimates in the following.

Theorem 5.5 *For each $f \in V$ and partition P as above, the linear finite-element interpolant f_0 of f with respect to P is characterized by* (5.12) *and it satisfies*

$$|\partial(f_0 - f)|_H \le |\partial f|_H \ , \tag{5.14}$$

and

$$|f_0 - f|_H \le 2\mu(P) |\partial f|_H \ . \tag{5.15}$$

If also $f \in H^2(G)$, then we have

$$|\partial(f_0 - f)|_H \le 2\mu(P) |\partial^2 f|_H \tag{5.16}$$

$$|f_0 - f|_H \le 4\mu(P)^2 |\partial^2 f|_H \ . \tag{5.17}$$

Proof: We need only to verify (5.16) and (5.17). Since $(f - f_0)(x_j) = 0$ for $0 \le j \le N+1$, we obtain for each $f \in H^2(G) \cap V$

$$|\partial(f_0 - f)|_H^2 = \sum_{j=0}^{N} \int_{x_j}^{x_{j+1}} (-\partial^2(f_0 - f))(f_0 - f)\, dx = (\partial^2 f, f_0 - f)_H \ ,$$

and thereby the estimate

$$|\partial(f_0 - f)|_H^2 \le |f_0 - f|_H |\partial^2 f|_H \ .$$

With (5.13) this gives (5.16) after dividing the factor $|\partial(f_0 - f)|_H$. Finally, (5.17) follows from (5.13) and (5.16).

Corollary 5.6 *For each h with $0 < h < 1$ let P_h be a partition of G with mesh $\mu(P_h) < h$, and define L_h to be the space of all linear finite-element function in $H^1(G)$ corresponding to the partition P_h. Then $\mathcal{L} \equiv \{L_h : 0 < h < 1\}$ satisfies the approximation assumption* (5.9) *with $k = 0$. The degree of $\mathcal{L}$ is* 1.

Finally we briefly consider the computations that are involved in implementing the Galerkin procedure (5.2) for one of the spaces L_h above. Let $P_h = \{x_0, x_1, \ldots, x_{N+1}\}$ be the corresponding partition and define ℓ_j to be the unique function in L_h which satisfies

$$\ell_j(x_i) = \begin{cases} 1 & \text{if } i = j \ , \\ 0 & \text{if } i \ne j \ , \end{cases} \qquad 0 \le i, j \le N+1 \ . \tag{5.18}$$

For each $f \in H^1(G)$, the interpolant f_0 is given by

$$f_0 = \sum_{j=0}^{N+1} f(x_j)\ell_j \ .$$

We use the basis (5.18) to write the problem in the form (5.5), and we must then invert the matrix $(a(\ell_i, \ell_j))$. Since $a(\cdot,\cdot)$ consists of integrals over G of products of ℓ_i and ℓ_j and their derivatives, and since any such product is identically zero when $|i - j| \ge 2$, it follows that the coefficient matrix is tridiagonal. It is also symmetric and positive-definite. There are efficient methods for inverting such matrices.

6 Approximation of Evolution Equations

We present here the Faedo-Galerkin procedure for approximating the solution of evolution equations of the types considered in Chapters IV, V and VI. As in the preceding section, the idea is to project a weak form of the problem onto a finite-dimensional subspace. We obtain thereby a system of ordinary differential equations whose solution approximates the solution of the original problem. In the applications to initial-boundary-value problems, this corresponds to a discretization of the space variable by a finite-element or spline approximation. We shall describe these semi-discrete approximation procedures, obtain estimates on the error that results from the approximation, and give the convergence rates that result from standard finite-element or spline approximations in the space variable. This program is carried out for first-order evolution equations and also for second-order evolution equations.

6.1

We first consider some first-order equations of the implicit type discussed in Section V.2. Let $\mathcal{M}$ be the Riesz map of the Hilbert space V_m with scalar-product $(\cdot,\cdot)_m$. Let V be a Hilbert space dense and continuously imbedded in V_m and let $\mathcal{L} \in \mathcal{L}(V, V')$. For a given $f \in C((0,\infty), V'_m)$ and $u_0 \in V_m$, we consider the problem of approximating a solution $u \in C([0,\infty), V_m) \cap C^1((0,\infty), V_m)$ of

$$\mathcal{M}u'(t) + \mathcal{L}u(t) = f(t) \ , \qquad t > 0 \ , \tag{6.1}$$

with $u(0) = u_0$. Since $\mathcal{M}$ is symmetric, such a solution satisfies

$$D_t(u(t), u(t))_m + 2\ell(u(t), u(t)) = 2f(t)(u(t)) \ , \qquad t > 0 \ , \tag{6.2}$$

where $\ell(\cdot,\cdot) : V \times V \to \mathbb{R}$ is the bilinear form associated with $\mathcal{L}$. This gives the identity

$$\|u(t)\|_m^2 + 2\int_0^t \ell(u(s), u(s))\, ds = \|u_0\|_m^2 + 2\int_0^t f(s)(u(s))\, ds \ , \qquad t > 0 \ , \tag{6.3}$$

involving the V_m norm $\|\cdot\|_m$ of the solution. Since the right side of (6.2) is bounded by $T\|f\|_{V'_m}^2 + T^{-1}\|u\|_m^2$ for any given $T > 0$, we obtain from (6.2)

$$D_t(e^{-t/T}\|u(t)\|_m^2) + e^{-t/T} 2\ell(u(t), u(t)) \le Te^{-t/T}\|f(t)\|_{V'_m}^2$$

and from this follows the a-priori estimate

$$\|u(t)\|_m^2 + 2\int_0^t \ell(u(s),u(s))\,ds \le e\|u_0\|^2 + Te\int_0^t \|f(s)\|_{V_m'}^2\,ds\,, \qquad 0 \le t \le T\,. \tag{6.4}$$

In the situations we consider below, $\mathcal{L}$ is monotone, hence, (6.4) gives an upper bound on the V_m norm of the solution.

In order to motivate the Faedo-Galerkin approximation, we note that a solution u of (6.1) satisfies

$$(u'(t),v)_m + \ell(u(t),v) = f(t)(v)\ , \qquad v \in V\ ,\ t > 0\ . \tag{6.5}$$

Since V is dense in V_m, (6.5) is actually equivalent to (6.1). Let S be a subspace of V. Then we consider the related problem of determining $u_s \in C([0,\infty),S) \cap C^1((0,\infty),S)$ which satisfies

$$(u_s'(t),v)_m + \ell(u_s(t),v) = f(t)(v)\ , \qquad v \in S\ ,\ t > 0 \tag{6.6}$$

and an initial condition to be specified.

Consider the case of S being a finite-dimensional subspace of V; let $\{v_1, v_2, \ldots, v_m\}$ be a basis for S. Then the solution of (6.6) is of the form

$$u_s(t) = \sum_{i=1}^m x_i(t) v_i$$

where $x(t) \equiv (x_1(t), x_2(t), \ldots, x_m(t))$ is in $C([0,\infty),\mathbb{R}^m) \cap C^1((0,\infty),\mathbb{R}^m)$, and (6.6) is equivalent to the system of ordinary differential equations

$$\sum_{i=1}^m (v_i,v_j)_m x_i'(t) + \sum_{i=1}^m \ell(v_i,v_j) x_i(t) = f(t)(v_j)\ , \qquad 1 \le j \le m\ . \tag{6.7}$$

The linear system (6.7) has a unique solution $x(t)$ with the initial condition $x(0)$ determined by $u_s(0) = \sum_{i=1}^m x_i(0) v_i$. (Note that the matrix coefficient of $x'(t)$ in (6.7) is symmetric and positive-definite, hence, nonsingular.) As in the preceding section, it is helpful to choose the basis functions so that most of the coefficients in (6.7) are zero. Special efficient computational techniques are then available for the resulting sparse system.

6.2

We now develop estimates on the error, $u(t)-u_s(t)$, in the situation of Theorem V.2.2. This covers the case of parabolic and pseudoparabolic equations. It will be shown that the error in the Faedo-Galerkin procedure for (6.1) is bounded by the error in the corresponding Rayleigh-Ritz-Galerkin procedure for the elliptic problem determined by the operator $\mathcal{L}$. Thus, we consider for each $t > 0$ the *$\mathcal{L}$-projection* of $u(t)$ defined by

$$u_\ell(t) \in S: \quad \ell(u_\ell(t), v) = \ell(u(t), v) \ , \qquad v \in S \ . \tag{6.8}$$

Theorem 6.1 *Let the real Hilbert spaces V and V_m, operators $\mathcal{M}$ and $\mathcal{L}$, and data u_0 and f be given as in Theorem V.2.2, and let S be a closed subspace of V. Then there exists a unique solution u of* (6.1) *with $u(0) = u_0$ and there exists a unique solution u_s of* (6.6) *for any prescribed initial value $u_s(0) \in S$. Assume $u \in C([0,\infty), V)$ and choose $u_s(0) = u_\ell(0)$, the $\mathcal{L}$-projection* (6.8) *of $u(0)$. Then we have the error estimate*

$$\|u(t) - u_s(t)\|_m \le \|u(t) - u_\ell(t)\|_m + \int_0^t \|u'(s) - u'_\ell(s)\|_m \, ds \ , \quad t \ge 0 \ . \tag{6.9}$$

Proof: The existence-uniqueness results are immediate from Theorem V.2.2, so we need only to verify (6.9). Note that $u(0) = u_0$ necessarily belongs to V, so (6.8) defines $u_\ell(0) = u_s(0)$. For any $v \in S$ we obtain from (6.5) and (6.6)

$$(u'(t) - u'_s(t), v)_m + \ell(u(t) - u_s(t), v) = 0 \ ,$$

so (6.8) gives the identity

$$(u'(t) - u'_\ell(t), v)_m = (u'_s(t) - u'_\ell(t), v)_m + \ell(u_s(t) - u_\ell(t), v) \ .$$

Setting $v = u_s(t) - u_\ell(t)$ and noting that $\mathcal{L}$ is monotone, we obtain

$$D_t \|u_s(t) - u_\ell(t)\|_m^2 \le 2\|u'(t) - u'_\ell(t)\|_m \|u_s(t) - u_\ell(t)\|_m \ .$$

The function $t \mapsto \|u_s(t) - u_\ell(t)\|_m$ is absolutely continuous, hence differentiable almost everywhere, and satisfies

$$D_t \|u_s(t) - u_\ell(t)\|_m^2 = 2\|u_s(t) - u_\ell(t)\|_m D_t \|u_s(t) - u_\ell(t)\|_m \ .$$

Let $Z = \{t > 0 : \|u_s(t) - u_\ell(t)\|_m = 0\}$. Clearly, for any $t \notin Z$ we have from above

$$D_t \|u_s(t) - u_\ell(t)\|_m \le \|u'(t) - u'_\ell(t)\|_m \ . \tag{6.10}$$

At an accumulation point of Z, the estimate (6.10) holds, since the left side is zero at such a point. Since Z has at most a countable number of isolated points, this shows that (6.10) holds at almost every $t > 0$. Integrating (6.10) gives the estimate

$$\|u_s(t) - u_\ell(t)\|_m \le \int_0^t \|u'(s) - u'_\ell(s)\|_m \, ds \ , \qquad t \ge 0 \ ,$$

from which (6.9) follows by the triangle inequality.

The fundamental estimate (6.9) shows that the error in the approximation procedure is determined by the error in the $\mathcal{L}$-projection (6.8) which is just the Rayleigh-Ritz-Galerkin procedure of Section 5. Specifically, when $u \in C^1((0,\infty), V)$ we differentiate (6.8) with respect to t and deduce that $u'_\ell(t) \in S$ is the $\mathcal{L}$-projection of $u'(t)$. This regularity of the solution u holds in both parabolic and pseudoparabolic cases.

We shall illustrate the use of the estimate (6.9) by applying it to a second order parabolic equation which is approximated by using a set of finite-element subspaces of degree one. Thus, suppose $\mathcal{S} \equiv \{S_h : h \in \mathcal{H}\}$ is a collection of closed subspaces of the closed subspace V of $H^1(G)$ and $\mathcal{S}$ is of degree 1; cf. Section 5.3. Let the continuous bilinear form $a(\cdot,\cdot)$ be V-elliptic and 0-regular; cf. Section III.6.4. Set $H = L^2(G) = H'$, so $\mathcal{M}$ is the identity, let $f \equiv 0$, and let $\ell(\cdot,\cdot) = a(\cdot,\cdot)$. If u is the solution of (6.1) and u_h is the solution of (6.6) with $S = S_h$, then the differentiability in $t > 0$ of these functions is given by Corollary IV.6.4 and their convergence at $t = 0^+$ is given by Exercise IV.7.8. We assume the form adjoint to $a(\cdot,\cdot)$ is 0-regular and obtain from (5.11) the estimates

$$\left.\begin{aligned} \|u(t) - u_\ell(t)\|_{L^2(G)} &\le c_2 h^2 \|Au(t)\|_{L^2(G)} \ , \\ \|u'(t) - u'_\ell(t)\|_{L^2(G)} &\le c_2 h^2 \|A^2 u(t)\|_{L^2(G)} \ , \qquad t > 0 \ . \end{aligned}\right\} \tag{6.11}$$

The a-priori estimate obtained from (6.3) shows that $|u(t)|_H$ is non-increasing and it follows similarly that $|Au(t)|_H$ is non-increasing for $t > 0$. Thus, if $u_0 \in D(A^2)$ we obtain from (6.9), and (6.11) the error estimate

$$\|u(t) - u_h(t)\|_{L^2(G)} \le c_2 h^2 \{\|Au_0\|_{L^2(G)} + t\|A^2 u_0\|_{L^2(G)}\} \ . \tag{6.12}$$

Although (6.12) gives the correct rate of convergence, it is far from optimal in the hypotheses assumed. For example, one can use estimates from Theorem IV.6.2 to play off the factors t and $\|Au'(t)\|_H$ in the second term of (6.12) and thereby relax the assumption $u_0 \in D(A^2)$. Also, corresponding estimates can be obtained for the non-homogeneous equation and faster convergence rates can be obtained if approximating subspaces of higher degree are used.

6.3

We turn now to consider the approximation of second-order evolution equations of the type discussed in Section VI.2. Thus, we let $\mathcal{A}$ and $\mathcal{C}$ be the respective Riesz maps of the Hilbert spaces V and W, where V is dense and continuously embedded in W, hence, W' is identified with a subspace of V'. Let $\mathcal{B} \in \mathcal{L}(V, V')$, $u_0 \in V$, $u_1 \in W$ and $f \in C((0,\infty), W')$. We shall approximate the solution $u \in C([0,\infty), V) \cap C^1((0,\infty), V) \cap C^1([0,\infty), W) \cap C^2((0,\infty), W)$ of

$$\mathcal{C}u''(t) + \mathcal{B}u'(t) + \mathcal{A}u(t) = f(t) \ , \qquad t > 0 \ , \tag{6.13}$$

with the initial conditions $u(0) = u_0$, $u'(0) = u_1$. Equations of this form were solved in Section VI.2 by reduction to an equivalent first-order system of the form (6.1) on appropriate product spaces. We recall here the construction, since it will be used for the approximation procedure. Define $V_m \equiv V \times W$ with the scalar product

$$([x_1, x_2], [y_1, y_2]) = (x_1, y_1)_V + (x_2, y_2)_W \ , \qquad [x_1, x_1], [y_1, y_1] \in V \times W \ ,$$

so $V_m' = V' \times W'$; the Riesz map $\mathcal{M}$ of V_m onto V_m' is given by

$$\mathcal{M}([x_1, x_2]) = [\mathcal{A}x_1, \mathcal{C}x_2] \ , \qquad [x_1, x_2] \in V_m \ .$$

Define $V_\ell = V \times V$ and $\mathcal{L} \in \mathcal{L}(V_\ell, V_\ell')$ by

$$\mathcal{L}([x_1, x_2]) = [-\mathcal{A}x_2, \mathcal{A}x_1 + \mathcal{B}x_2] \ , \qquad [x_1, x_2] \in V_\ell \ .$$

Then Theorem VI.2.1 applies if $\mathcal{B}$ is monotone to give existence and uniqueness of a solution $w \in C^1([0,\infty), V_m)$ of

$$\mathcal{M}w'(t) + \mathcal{L}w(t) = [0, f(t)] \ , \qquad t > 0 \tag{6.14}$$

with $w(0) = [u_0, u_1]$ and $f \in C^1([0,\infty), W')$ given so that $u_0, u_1 \in V$ with $\mathcal{A}u_0 + \mathcal{B}u_1 \in W'$. The solution is given by $w(t) = [u(t), u'(t)]$, $t \geq 0$;

from the inclusion $[u, u'] \in C^1([0,\infty), V \times W)$ and (6.14) we obtain $[u, u'] \in C^1([0,\infty), V \times V)$. From (6.4) follows the a-priori estimate

$$\|u(t)\|_V^2 + \|u'(t)\|_W^2 + 2\int_0^t \mathcal{B}u'(s)(u'(s))\,ds$$
$$\leq e(\|u_0\|_V^2 + \|u_1\|_W^2) + Te\int_0^t \|f(s)\|_{W'}^2\,ds\ , \qquad 0 \leq t \leq T\ ,$$

on a solution $w(t) = [u(t), u'(t)]$ of (6.14).

The Faedo-Galerkin approximation procedure for the second-order equation is just the corresponding procedure for (6.14) as given in Section 6.1. Thus, if S is a finite-dimensional subspace of V, then we let w_s be the solution in $C^1([0,\infty), S \times S)$ of the equation

$$(w_s'(t), v)_m + \ell(w(t), v) = [0, f(t)](v)\ , \qquad v \in S \times S\ ,\ t > 0\ , \qquad (6.15)$$

with an initial value $w_s(0) \in S \times S$ to be prescribed below. If we look at the components of $w_s(t)$ we find from (6.15) that $w_s(t) = [u_s(t), u_s'(t)]$ for $t > 0$ where $u_s \in C^2([0,\infty), S)$ is the soluton of

$$(u_s''(t), v)_W + b(u_s'(t), v) + (u_s(t), v)_V = f(t)(v)\ , \quad v \in S\ ,\ t > 0\ . \quad (6.16)$$

Here $b(\cdot,\cdot)$ denotes the bilinear form on V corresponding to $\mathcal{B}$. As in Section 6.1, we can choose a basis for S and use it to write (6.16) as a system of m ordinary differential equations of second order. Of course this system is equivalent to a system of $2m$ equations of first order as given by (6.15), and this latter system may be the easier one in which to do the computation.

6.4

Error estimates for the approximation of (6.13) by the related (6.16) will be obtained in a special case by applying Theorem 6.1 directly to the situation described in Section 6.3. Note that in the derivation of (6.9) we needed only that $\mathcal{L}$ is monotone. Since $\mathcal{B}$ is monotone, the estimate (6.9) holds in the present situation. This gives an error bound in terms of the $\mathcal{L}$-projection $w_\ell(t) \in S \times S$ of the solution $w(t)$ of (6.14) as defined by

$$\ell(w_\ell(t), v) = \ell(w(t), v)\ , \qquad v \in S \times S\ . \qquad (6.17)$$

The bilinear form $\ell(\cdot,\cdot)$ is not coercive over V_ℓ so we might not expect $w_\ell(t) - w(t)$ to be small. However, in the special case of $\mathcal{B} = \varepsilon\mathcal{A}$ for some $\varepsilon \geq 0$ we find that (6.17) is equivalent to a pair of similar identities in the component spaces. That is, if $e(t) \equiv w(t) - w_\ell(t)$ denotes the error in the $\mathcal{L}$-projection, and if $e(t) = [e_1(t), e_2(t)]$, then (6.17) is equivalent to

$$(e_j(t), v)_V = 0 \ , \qquad v \in S \ , \ j = 1, 2 \ . \tag{6.18}$$

Thus, if we write $w_\ell(t) = [u_\ell(t), v_\ell(t)]$, we see that $u_\ell(t)$ is the V-projection of $u(t)$ on S and $v_\ell(t) = u'_\ell(t)$ is the projection of $u'(t)$ on S. It follows from these remarks that we have

$$\|u(t) - u_\ell(t)\|_V \leq \inf\{\|u(t) - v\|_V : v \in S\} \tag{6.19}$$

and corresponding estimates on $u'(t) - u'_\ell(t)$ and $u''(t) - u''_\ell(t)$. Our approximation results for (6.13) can be summarized as follows.

Theorem 6.2 *Let the Hilbert spaces V and W, operators $\mathcal{A}$ and $\mathcal{C}$, and data u_0, u_1 and f be given as in Theorem VI.2.1. Suppose furthermore that $\mathcal{B} = \varepsilon\mathcal{A}$ for some $\varepsilon \geq 0$ and that S is a finite-dimensional subspace of V. Then there exists a unique solution $u \in C^1([0,\infty), V) \cap C^2([0,\infty), W)$ of* (6.13) *with $u(0) = u_0$ and $u'(0) = u_1$; and there exists a unique solution $u_s \in C^2([0,\infty), S)$ of* (6.16) *with initial data determined by*

$$(u_s(0) - u_0, v)_V = (u'_s(0) - u_1, v)_V = 0 \ , \qquad v \in S \ .$$

We have the error estimate

$$\begin{aligned}
&(\|u(t) - u_s(t)\|_V^2 + \|u'(t) - u'_s(t)\|_W^2)^{1/2} \\
&\quad \leq (\|u(t) - u_\ell(t)\|_V^2 + \|u'(t) - u'_\ell(t)\|_W^2)^{1/2} \\
&\qquad + \int_0^t (\|u'(s) - u'_\ell(s)\|_V^2 + \|u''(s) - u''_\ell(s)\|_W^2)^{1/2}\, ds \ , \qquad t \geq 0
\end{aligned} \tag{6.20}$$

where $u_\ell(t) \in S$ is the V-projection of $u(t)$ defined by

$$(u_\ell(t), v)_V = (u(t), v)_V \ , \qquad v \in S \ .$$

Thus (6.19) *holds and provides a bound on* (6.20).

Finally we indicate how the estimate (6.20) is applied with finite-element or spline function spaces. Suppose $\mathcal{S} = \{S_h : h \in \mathcal{H}\}$ is a collection of finite-dimensional subspaces of the closed subspace V of $H^1(G)$. Let $k+1$ be the degree of $\mathcal{S}$ which satisfies the approximation assumption (5.9). The scalar-product on V is equivalent to the $H^1(G)$ scalar-product and we assume it is k-regular on V. For each $h \in \mathcal{H}$ let u_h be the solution of (6.16) described above with $S = S_h$, and suppose that the solution u satisfies the regularity assumptions $u, u' \in L^\infty([0,T], H^{k+2}(G))$ and $u'' \in L^1([0,T], H^{k+2}(G))$. Then there is a constant c_0 such that

$$\begin{aligned} &(\|u(t) - u_h(t)\|_V^2 + \|u'(t) - u_h'(t)\|_h^2)^{1/2} \\ &\qquad \leq c_0 h^{k+1} , \qquad h \in \mathcal{H} \ , \ 0 \leq t \leq T \ . \end{aligned} \tag{6.21}$$

The preceding results apply to wave equations (cf. Section VI.2.1), viscoelasticity equations such as VI.(2.9), and Sobolev equations (cf. Section VI.3).

Exercises

1.1. Show that a solution of the Neumann problem $-\Delta_n u = F$ in G, $\partial u/\partial v = 0$ on ∂G is a $u \in H^1(G)$ at which the functional (1.3) attains its minimum value.

2.1. Show that $F : K \to \mathbb{R}$ is weakly lower-semi-continuous at each $x \in K$ if and only if $\{x \in V : F(x) \leq a\}$ is weakly closed for every $a \in \mathbb{R}$.

2.2. In the proof of Theorem 2.3, show that $\varphi'(t) = F'(y + t(x-y))(x-y)$.

2.3. In the proof of Theorem 2.7, verify that M is closed and convex.

2.4. Prove Theorem 2.9.

2.5. Let F be G-differentiable on K. If F' is strictly monotone, prove directly that (2.5) has at most one solution.

2.6. Let G be bounded and open in $\mathbb{R}^n$ and let $F : G \times \mathbb{R} \to \mathbb{R}$ satisfy the following: $F(\cdot, u)$ is measurable for each $u \in \mathbb{R}$, $F(x, \cdot)$ is absolutely continuous for almost every $x \in G$, and the estimates

$$|F(x,u)| \leq a(x) + b|u|^2 \quad , \quad |\partial_u F(x,u)| \leq c(x) + b|u|$$

hold for all $u \in \mathbb{R}$ and a.e. $x \in G$, where $a(\cdot) \in L^1(G)$ and $c(\cdot) \in L^2(G)$.

(a) Define $E(u) = \int_G F(x, u(x))\,dx$, $u \in L^2(G)$, and show

$$E'(u)(v) = \int_G \partial_u F(x, u(x))v(x)\,dx \ , \qquad u, v \in L^2(G) \ .$$

(b) Show E' is monotone if $\partial_u F(x, \cdot)$ is non-decreasing for a.e. $x \in G$.

(c) Show E' is coercive if for some $k > 0$ and $c_0(\cdot) \in L^2(G)$ we have

$$\partial_u F(x, u) \cdot u \geq k|u|^2 - c_0(x)|u| \ ,$$

for $u \in \mathbb{R}$ and a.e. $x \in G$.

(d) State and prove some existence theorems and uniqueness theorems for boundary value problems containing the semi-linear equation

$$-\Delta_n u + f(x, u(x)) = 0 \ .$$

2.7. Let G be bounded and open in $\mathbb{R}^n$. Suppose the function $F : G \times \mathbb{R}^{n+1} \to \mathbb{R}$ satisfies the following: $F(\cdot, \hat{u})$ is measurable for $\hat{u} \in \mathbb{R}^{n+1}$, $F(x, \cdot) : \mathbb{R}^{n+1} \to \mathbb{R}$ is (continuously) differentiable for a.e. $x \in G$, and the estimates

$$|F(x, \hat{u})| \leq a(x) + b\sum_{j=0}^{n} |u_j|^2 \quad , \quad |\partial_k F(x, \hat{u})| \leq c(x) + b\sum_{j=0}^{n} |u_j|$$

as above for every k, $0 \leq k \leq n$, where $\partial_k = \frac{\partial}{\partial u_k}$.

(a) Define $E(u) = \int_G F(x, u(x), \nabla u(x))\,dx$, $u \in H^1(G)$, and show

$$E'(u)(v) = \int_G \sum_{j=0}^{n} \partial_j F(x, u, \nabla u)\partial_j v(x)\,dx \ , \qquad u, v \in H^1(G) \ .$$

(b) Show E' is monotone if

$$\sum_{j=0}^{n} (\partial_j F(x, u_0, u_1, \ldots, u_n) - \partial_j F(x, v_0, v_1, \ldots, v_n))(u_j - v_j) \geq 0$$

for all $\hat{u}, \hat{v} \in \mathbb{R}^{n+1}$ and a.e. $x \in G$.

(c) Show E' is coercive if for some $k > 0$ and $c_0(\cdot) \in L^2(G)$

$$\sum_{j=0}^{n} \partial_j F(x, \hat{u})u_j \geq k\sum_{j=0}^{n} |u_j|^2 - c_0(x)\sum_{j=0}^{n} |u_j|$$

for $\hat{u} \in \mathbb{R}^{n+1}$ and a.e. $x \in \mathbb{R}^n$.

(d) State and prove an existence theorem and a uniqueness theorem for a boundary value problem containing the nonlinear equation

$$\sum_{j=0}^{n} \partial_j F_j(x, u, \nabla u) = f(x) \ .$$

3.1. Prove directly that (3.4) has at most one solution when $a(\cdot, \cdot)$ is (strictly) positive.

3.2. Give an example of a stretched membrane (or string) problem described in the form (3.6). Specifically, what does g represent in this application?

4.1. Show the following optimal control problem is described by the abstract setting of Section 4.1: find an admissible control $u \in U_{\text{ad}} \subset L^2(G)$ which minimizes the function

$$J(u) = \int_G |y(u) - w|^2 \, dx + c \int_G |u|^2 \, dx$$

subject to the state equations

$$\begin{cases} -\Delta_n y = F + u & \text{in } G \ , \\ y = 0 & \text{on } \partial G \ . \end{cases}$$

Specifically, identify all the spaces and operators in the abstract formulation.

4.2. Give sufficient conditions on the data above for existence of an optimal control. Write out the optimality system (4.10) for cases analogous to Sections 4.5 and 4.6.

5.1. Write out the special cases of Theorems 5.1 and 5.2 as they apply to the boundary value problem

$$\begin{cases} -\partial(p(x)\partial u(x)) + q(x)u(x) = f(x) \ , & 0 < x < 1 \ , \\ u(0) = u(1) = 0 \ . \end{cases}$$

Give the algebraic problem (5.5) and error estimates that occur when the piecewise-linear functions of Section 5.4 are used.

5.2. Repeat the above for the boundary value problem

$$\begin{cases} -\partial(p(x)\partial u(x)) + q(x)u(x) = f(x) \ , \\ u'(0) = u'(1) = 0 \ . \end{cases}$$

(Note that the set K and subspaces are not exactly as above.)

5.3. We describe an *Hermite interpolation* by piecewise-cubics. Let the interval G and partition P be given as in Section 5.4. Let $V \leq H^2(G)$ and define

$$K = \{v \in V : v(x_j) = v'(x_j) = 0 \ , \quad 0 \leq j \leq N+1\} \ .$$

(a) Let $f \in V$ and define $F(v) = (\frac{1}{2})|\partial^2(v-f)|_{L^2(G)}$. Show there is a unique $u_f \in K : (\partial^2(u_f - f), \partial^2 v)_{L^2(G)} = 0$, $v \in K$.

(b) Show there exists a unique $f_0 \in H^2(G)$ for which f_0 is a cubic polynomial on each $[x_j, x_{j+1}]$, $f_0(x_j) = f(x_j)$ and $f_0'(x_j) = f'(x_j)$ for $j = 0, 1, \ldots, N+1$.

(c) Construct a corresponding family of subspaces as in Theorem 5.4 and show it is of degree 3.

(d) Repeat exercise 5.1 using this family of approximating subspaces.

5.4. Repeat exercise 5.3 but with $V = H_0^2(G)$ and

$$K = \{v \in V : v(x_j) = 0 \ , \quad 0 \leq j \leq N+1\} \ .$$

Show that the corresponding *Spline interpolant* is a piecewise-cubic, $f_0(x_j) = f(x_j)$ for $0 \leq j \leq N+1$, and f_0 is in $C^2(G)$.

6.1. Describe the results of Sections 6.1 and 6.2 as they apply to the problem

$$\begin{cases} \partial_t u(x,t) - \partial_x(p(x)\partial_x u(x,t)) = F(x,t) \ , \\ u(0,t) = u(1,t) = 0 \ , \\ u(x,0) = u_0(x) \ . \end{cases}$$

Use the piecewise-linear approximating subspaces of Section 5.4.

6.2. Describe the results of Sections 6.3 and 6.4 as they apply to the problem

$$\begin{cases} \partial_t^2 u(x,t) - \partial_x(p(x)\partial_x u(x,t)) = F(x,t) \ , \\ u(0,t) = u(1,t) = 0 \ , \\ u(x,0) = u_0(x) \ , \quad \partial_t u(x,0) = u_1(x) \ . \end{cases}$$

Use the subspaces of Section 5.4.

Chapter VIII

Suggested Readings

Chapter I
This material is covered in almost every text on functional analysis. We mention specifically references [22], [25], [47].

Chapter II

Our definition of distribution in Section 1 is inadequate for many purposes. For the standard results see any one of [8], [24], [25]. For additional information on Sobolev spaces we refer to [1], [3], [19], [33], [36].

Chapter III

Linear elliptic boundary value problems are discussed in the references [2], [3], [19], [33], [35], [36] by methods closely related to ours. See [22], [24], [43], [47] for other approaches. For basic work on nonlinear problems we refer to [5], [8], [32], [41].

Chapter IV

We have only touched on the theory of semigroups; see [6], [19], [21], [23], [27], [47] for additional material. Refer to [8], [19], [28], [30] for hyperbolic problems and [8], [26], [29], [35] for hyperbolic systems. Corresponding results for nonlinear problems are given in [4], [5], [8], [32], [34], [41], [47].

Chapter V and VI

The standard reference for implicit evolution equations is [9]. Also see [30] and [32], [41] for related linear and nonlinear results, respectively.

Chapter VII

For extensions and applications of the basic material of Section 2 see [8], [10], [17], [39], [45]. Applications and theory of variational inequalities are presented in [16], [18], [32]; their numerical approximation is given in [20]. See [31] for additional topics in optimal control. The theory of approximation of partial differential equations is given in references [3], [11], [37], [40], [42]; also see [10], [14].

Additional Topics

We have painfully rejected the temptation to pursue many interesting topics; each of them deserves attention. A few of these topics are improperly posed problems [7], [38], function-theoretic methods [12], bifurcation [15], fundamental solutions [24], [43], scattering theory [29], the transposition method [33], non-autonomous evolution equations [5], [8], [9], [19], [27], [30], [34], [47], and singular problems [9].

Classical treatments of partial differential equations of elliptic and hyperbolic type are given in the treatise [13] and the canonical parabolic equation is discussed in [46]. These topics are similarly presented in [44] together with derivations of many initial and boundary value problems and their applications.

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